ON NESTED SEQUENTS FOR CONSTRUCTIVE MODAL LOGICS

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ABSTRACT. We present deductive systems for various modal logics that can be obtained from the constructive variant of the normal modal logic CK by adding combinations of the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$. This includes the constructive variants of the standard modal logics $\mathsf{K4}$, $\mathsf{S4}$, and $\mathsf{S5}$. We use for our presentation the formalism of nested sequents and give a syntactic proof of cut elimination.

1. INTRODUCTION

The modal logic K is obtained from classical propositional logic by incorporating two unary operators, or *modalities*, $\Box$ and $\Diamond$, and adding the k-axiom, $\Box(A \supset B) \supset (\Box A \supset \Box B)$, to dictate the interaction between the modalities and propositional connectives. The behaviour of the $\Diamond$ modality is then determined by enforcing that it is the De Morgan dual of $\Box$. Along with this axiom there is the *necessitation* rule, saying that if A is a theorem of K then so is $\Box A$. Informally, $\Box$ is often interpreted as "necessarily" and $\Diamond$ as "possibly". Notice that interaction with other propositional connectives is determined by the adequacy of $\{\supset, \bot\}$ in classical logic.

In the intuitionistic setting, however, one must define the behaviour of $\Box$ and $\Diamond$ independently, in the absence of De Morgan duality. Consequently, it is not enough to just add the standard k-axiom, which makes no mention of the $\Diamond$-modality, and so some classical consequences of k must be added to formulate an intuitionistic version of K. To this end there seems to be no canonical choice, and many different intuitionistic versions of K have been proposed, e.g., [Fit48, Pra65, Ser84, PS86, Sim94, BdP00, PD01] (for a survey see [Sim94]). However, in the current literature, two variants prevail; the first, known as *intuitionistic* K, adds the following five axioms, along with the necessitation rule, to intuitionistic propositional logic:

$$\begin{array}{ll} \mathsf{k}_1: \ \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: \ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: \ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: \ (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & \mathsf{k}_5: \ \Diamond \bot \supset \bot \end{array} \tag{1.1}$$

It was originally proposed in [Ser84, PS86] and studied in detail in [Sim94]; more recent work can be found in [GS10, Str13].

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The second variant, known as *constructive* K, rejects the axioms $\mathsf{k}_3, \mathsf{k}_4, \mathsf{k}_5$ and incorporates only k_1 and k_2. This choice of axioms dates back to [Pra65][1], and its proof theory was investigated, for example, in [BdP00, HP07, MS11], while the semantics of it and some extensions was studied in [FM97] and [Koj12].

To gain intuition about the difference between the two variants, let us have a look at their standard Kripke semantics. A model of intuitionistic modal logic is described by a 4-tuple $(W, \leq, R, \mathsf{I})$ with

- a non-empty set of possible worlds W pre-ordered in $\leq$.
- accessibility relation $R \subseteq W \times W$ satisfying
  - (i) For any $w, v, v' \in W$, if wRv and $v \leq v'$, then there exists a $w' \in W$ such that $w \leq w'$ and $w'Rv'$, and
  - (ii) For any $w, w', v \in W$, if $w \leq w'$ and wRv, then there exists a $v' \in W$ such that $w'Rv'$ and $v \leq v'$.
- a function $\mathsf{I}\colon W \to 2^{\mathcal{P}}$, where $\mathcal{P} = \{a, b, c, \ldots\}$ denotes the set of propositional letters, such that for any $w, w' \in W$, if $w \leq w'$ then $\mathsf{I}(w) \subseteq \mathsf{I}(w')$.

Note that (i) and (ii) ensure monotonicity of R over the order of $\leq$. In comparison, a model of constructive modal logic decouples the accessibility relation R from $\leq$. It assumes a set of fallible worlds $\dot{\bot}$ as a sub-set of W; such that $\dot{\bot}$ is closed under $\leq$ and R, *i.e.* whenever $w \in \dot{\bot}$ and wRw_1 or $w \leq w_1$ we also have $w_1 \in \dot{\bot}$. This is much weaker a condition on R than (i) and (ii). The semantics of $\Box$, $\Diamond$ and $\bot$ in the constructive setting are,

- $w \models \Box A$ iff $\forall w \leq w' \, \forall v'. w'Rv'$ implies $v' \models A$.
- $w \models \Diamond A$ iff $\forall w \leq w' \exists v'. w'Rv'$ and $v' \models A$.
- $w \models \bot$ iff $w \in \dot{\bot}$.
- $w \models A$ if $w \in \dot{\bot}$.

The semantics for the other connectives is defined in the same way as for the intuitionistic modal logic, i.e., the standard intuitionistic logic Kripke semantics:

- $w \models a$ iff $a \in \mathsf{I}(w)$.
- $w \models A \wedge B$ iff $w \models A$ and $w \models B$.
- $w \models A \vee B$ iff $w \models A$ or $w \models B$.
- $w \models A \supset B$ iff $\forall w' \in W. w \leq w'$ and $w' \models A$ imply $w' \models B$.

Then we have a counter-model to each of $\mathsf{k}_3, \mathsf{k}_4, \mathsf{k}_5$:

- k_3: $W = \{w_0, w_1, u_0, v_1\}, w_0 \leq w_1, w_0 R u_0, w_1 R v_1, \mathsf{I}(w_0) = \mathsf{I}(w_1) = \emptyset,$ $\mathsf{I}(u_0) = a_1, \mathsf{I}(v_1) = a_2$. We have $w_0 \models \Diamond(a_1 \vee a_2)$ and $w_0 \not\models \Diamond a_1 \vee \Diamond a_2$.
- k_4: $W = \{w_0, w_1, u_0, u_1, v_0, v_1\}, w_0 \leq w_1, u_0 \leq u_1, v_0 \leq v_1, w_0 R u_0,$ $w_0 R v_0, w_1 R u_1, \mathsf{I}(w_0) = \mathsf{I}(w_1) = \emptyset, \mathsf{I}(u_0) = \mathsf{I}(u_1) = \{a_1, a_2\}, \mathsf{I}(v_0) = \{a_2\},$ $\mathsf{I}(v_1) = \{a_1\}$. We have $w_0 \models \Diamond a_1 \supset \Box a_2$ and $w_0 \not\models \Box(a_1 \supset a_2)$.
- k_5: $W = \{w_0, u_0\}, w_0 R u_0, \mathsf{I}(w_0) = \emptyset, u_0 \in \dot{\bot}$. We have $w_0 \models \Diamond \bot$ and $w_0 \not\models \bot$.

Note that the counter-models for k_3 and k_4 cannot exist if we add conditions (i) and (ii) mentioned above, and the counter-models for k_5 relies on $\dot{\bot}$, which is not present in the semantics for intuitionistic modal logic.

However, we will not go into further detail here, and refer the reader to [MS11] for a more thorough semantic analysis of the difference between intuitionistic K and constructive K.

[1] We point out that some versions of K intermediate to these two variants have also been considered, for example the variant with $\mathsf{k}_1, \mathsf{k}_2$, and k_5 in [Wij90].

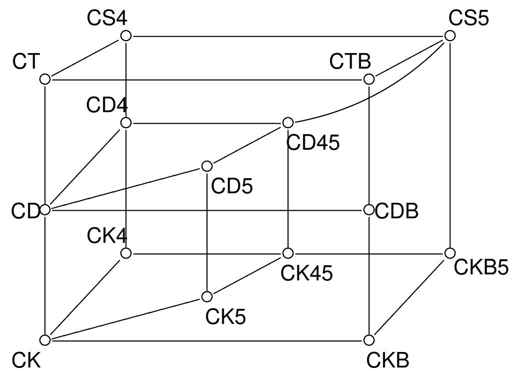

Figure 1: The constructive "modal cube"

Our work here is concerned with the proof theory of constructive K, denoted CK and its various extensions with other common modal axioms. Like the classical and intuitionistic variants, we consider the five axioms below:

$$\begin{array}{rl} \mathsf{d}\colon & \Box A \supset \Diamond A \\ \mathsf{t}\colon & (A \supset \Diamond A) \wedge (\Box A \supset A) \\ \mathsf{b}\colon & (A \supset \Box\Diamond A) \wedge (\Diamond \Box A \supset A) \end{array} \qquad \begin{array}{rl} \mathsf{4}\colon & (\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A) \\ \mathsf{5}\colon & (\Diamond A \supset \Box\Diamond A) \wedge (\Diamond \Box A \supset \Box A) \end{array} \tag{1.2}$$

A priori, this gives us 32 different logics, but as in classical modal logic some of them coincide, so that we get 15 logics,[2] which are depicted in Figure 1.

In this work we attempt to give a unified cut-elimination for all logics obtained using the framework of *nested sequents* [Kas94, GPT09, Brü09, Str13, Fit14], a generalisation of Gentzen's sequent calculus that allows sequents to occur within sequents. This approach has previously been successful for the classical modal cube in [Brü09] and the intuitionistic modal cube in [Str13] but, perhaps surprisingly, the step from intuitionistic to constructive appears more involved than the one from classical to intuitionistic. While the cut-elimination proofs in [Brü09] and [Str13] are more or less the same, we seem to require a different method in the constructive setting. The reasons are that certain formulations of some logical rules are no longer sound, and that we need an explicit contraction rule, along with other structural rules that further complicate the process of cut-elimination.

Nonetheless we manage to obtain cut-elimination for the logics CK, $\mathsf{CK4}$, $\mathsf{CK45}$, CD, $\mathsf{CD4}$, $\mathsf{CD45}$, CT, $\mathsf{CS4}$, and $\mathsf{CS5}$, but conjecture that our systems admit cut for all logics in the cube. To our knowledge, previously only the logics CK, CT, $\mathsf{CK4}$, and $\mathsf{CS4}$ have received analogous proof theoretic treatment [BdP00, HP07, MS11].

We point out an interesting observation that the b-axiom entails k_3 and k_5. While this is likely already known to many in the community we could not find this result stated in the literature, and so it is pertinent to raise it here. This arguably questions the "constructiveness" of logics including b, and so the inclusion of such logics in the cube itself, but such considerations are beyond the scope of this work.

Several previous attempts to deal with the proof theory of constructive modal logic have appeared, however, the fundamental data structures of such calculi all seem to be special cases of nested sequents. For example, the 2-sequents of [Mas92, Mas93] are a form of nested

[2]That there are at least 15 is inherited from the classical setting, and verifying that the classical equivalences hold is by inspection of the classical proofs.

sequent where no tree-branching is allowed. It is not clear how the 2-sequent approach, while successful for deontic logic, could be adapted for the various constructive logics, or even CK, as pointed out by Wansing in [Wan94]. Also the sequents of [MS11, MS14] can be seen as a special case of nested sequents where no tree-branching is allowed, but they have a richer data structure than 2-sequents because they include a focus.

For applications of the family of the constructive modal logics, the extended Curry-Howard correspondence (which for modal logics is a relatively recent investigation) was studied for $\mathsf{CS4}$ [AMdPR01, MS14]. The constructive $\Box$ operator is known to capture staged computation [DP96, aBMTS99], and such logics are also used for the study of contexts [MdP05, MS14]. We also point out that there are many logics of interest that are proper extensions of CK but not of intuitionistic K, e.g. $\mathsf{CS4}$ and PLL; a more detailed discussion of such logics can be found in [FM97].

2. Preliminaries on Nested Sequents

In order to present a nested sequent system for CK, we first need to define the notion of a nested sequent structure. For this, we recall the basic notions from [Str13], with slight modifications in notation tailored for the current setting. Let $a, b, c, \ldots$ denote propositional variables and define formulas $A, B, C, \ldots$ of constructive modal logic by the following grammar:

$$A ::= a \mid \bot \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$$

As shorthand we write $\top$ for $\bot \supset \bot$ and we omit parentheses whenever it is not ambiguous.

A (nested) sequent is a tree whose nodes are multisets of formulas tagged with a polarity. There are two polarities, *input* (intuitively as if on the left of the turnstile in the conventional sequent calculus), denoted by a $\bullet$ superscript, and *output* (intuitively as if on the right of the turnstile in the conventional sequent calculus), denoted by a $\circ$ superscript. Formally we define *LHS sequents*, denoted Φ, and *RHS sequents*, denoted Ψ, as follows,

$$\Phi ::= \emptyset \mid A^\bullet \mid [\Phi] \mid \Phi, \Phi \qquad\qquad \Psi ::= A^\circ \mid [\Phi, \Psi] \tag{2.1}$$

and a *full sequent* is a structure of the form Φ, Ψ. We assume that associativity and commutativity of the comma ‘,’ is implicit in our systems, and that $\emptyset$ acts as its unit.

This definition entails that exactly one formula in a sequent has output polarity, and all others have input polarity. We use capital Greek letters $\Gamma, \Delta, \Sigma, \ldots$ to denote arbitrary sequents, LHS, RHS or full, and may decorate them with a $\bullet$ or $\circ$ superscript to indicate that they are LHS or RHS respectively.

The *corresponding formula* of a sequent is defined inductively as follows,

$$fm(A^\bullet) = fm(A^\circ) = A \ , \ fm([\Phi]) = \Diamond fm(\Phi) \ , \ fm([\Phi, \Psi]) = \Box(fm(\Phi, \Psi))$$
$$fm(\emptyset) = \top \ , \ fm(\Phi_1, \Phi_2) = fm(\Phi_1) \wedge fm(\Phi_2) \ , \ fm(\Phi, \Psi) = fm(\Phi) \supset fm(\Psi)$$

A *context*, denoted by $\Gamma\{\ \}$, is a sequent with a hole $\{\ \}$ taking the place of a formula; $\Gamma\{\Delta\}$ is the sequent obtained from $\Gamma\{\ \}$ by replacing the occurrence of $\{\ \}$ by Δ. Note that, for this to form a full sequent, $\Gamma\{\ \}$ and Δ must have the correct form. We distinguish two kinds of contexts: an *output context* is one that results in a full sequent when its hole is filled with a RHS sequent, and an *input context* analogously for a LHS sequent. This is clarified by the following example, taken from [Str13].

Example 2.1. Let $\Gamma_1\{\ \} = C^\bullet, [\{\ \}, [B^\bullet, C^\bullet]]$ and $\Gamma_2\{\ \} = C^\bullet, [\{\ \}, [B^\bullet, C^\circ]]$. Now let $\Delta_1 = A^\bullet, [B^\circ]$ and $\Delta_2 = A^\bullet, [B^\bullet]$. Then $\Gamma_1\{\Delta_2\}$ and $\Gamma_2\{\Delta_1\}$ are not well-formed full sequents, because the former would contain no output formula, and the latter would contain two. However, we can form the full sequents,

$$\Gamma_1\{\Delta_1\} = C^\bullet, [A^\bullet, [B^\circ], [B^\bullet, C^\bullet]] \quad \text{and} \quad \Gamma_2\{\Delta_2\} = C^\bullet, [A^\bullet, [B^\bullet], [B^\bullet, C^\circ]]$$

whose corresponding formulas, respectively, are:

$$C \supset \Box(A \wedge \Diamond(B \wedge C) \supset \Box B) \quad \text{and} \quad C \supset \Box(A \wedge \Diamond B \supset \Box(B \supset C))$$

Observation 2.2. Every output context $\Gamma\{\ \}$ is of the shape,

$$\Gamma_1^\bullet, [\Gamma_2^\bullet, [\ldots, [\Gamma_n^\bullet, \{\ \}] \ldots]] \tag{2.2}$$

for some $n \geq 0$. Filling the hole of an output context with a RHS or full sequent yields a full sequent, and filling it with a LHS sequent yields a LHS sequent. Every input context $\Gamma\{\ \}$ is of the shape,

$$\Gamma'\{\Lambda\{\ \}, \Pi^\circ\} \tag{2.3}$$

where $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ are output contexts (i.e., are of the shape (2.2) above). Note that $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ and Π are uniquely determined by the position of the hole $\{\ \}$ in $\Gamma\{\ \}$.

We can choose to fill the hole of a context $\Gamma\{\ \}$ with nothing, denoted by $\Gamma\{\emptyset\}$, which means we simply remove the $\{\ \}$. In Example 2.1 above, $\Gamma_1\{\emptyset\} = C^\bullet, [[B^\bullet, C^\bullet]]$ is a LHS sequent and $\Gamma_2\{\emptyset\} = C^\bullet, [[B^\bullet, C^\circ]]$ is a full sequent. More generally, whenever $\Gamma\{\emptyset\}$ is a full sequent, then $\Gamma\{\ \}$ is an input context.

Definition 2.3. For every input context $\Gamma\{\ \} = \Gamma'\{\Lambda\{\ \}, \Pi^\circ\}$, we define its *output pruning* $\Gamma^\Downarrow\{\ \}$ to be the context $\Gamma'\{\Lambda\{\ \}\}$, i.e., the same context with the subtree containing the unique output formula and sharing the same root as $\{\ \}$ removed. Thus, $\Gamma^\Downarrow\{\ \}$ is an output context.

If $\Gamma\{\ \}$ is already an output context then $\Gamma^\Downarrow\{\ \} = \Gamma\{\ \}$. For every full sequent $\Delta = \Lambda^\bullet, \Pi^\circ$, we define $\Delta^\Downarrow$ to be its LHS-sequent Λ. For a LHS sequent Δ, we define $\Delta^\Downarrow = \Delta$.

In Example 2.1 above, $\Gamma_1^\Downarrow\{\ \} = C^\bullet, [\{\ \}, [B^\bullet, C^\bullet]]$ and $\Gamma_2^\Downarrow\{\ \} = C^\bullet, [\{\ \}]$ and $(\Gamma_1\{\emptyset\})^\Downarrow = C^\bullet, [[B^\bullet, C^\bullet]]$ and $(\Gamma_2\{\emptyset\})^\Downarrow = C^\bullet$. In particular note that $\Gamma_1^\Downarrow\{A^\circ\} = C^\bullet, [A^\circ, [B^\bullet, C^\bullet]]$, which is not the same as $(\Gamma_1\{A^\circ\})^\Downarrow = C^\bullet$.

3. Nested Sequent Systems for CK and its Variants

We present the nested sequent system NCK for CK in Figure 2. The rules are similar to the corresponding rules for intuitionistic modal logic in [Str13] and classical modal logic in [Brü09], but there are some subtle but crucial differences:

- In [Str13] and [Brü09] additive versions of $\supset^\bullet$ and $\Box^\bullet$ were given rather than incorporating an explicit contraction rule in the system. While these were essentially design choices in the previous works, here it is necessary to make contraction explicit since our treatment of the b-axiom does not allow us to show the admissibility of contraction, further explained below. Consequently, our cut elimination proof differs significantly from the ones in [Str13] and [Brü09].

$$\bot^\bullet \frac{}{\Gamma\{\bot^\bullet, \Pi^\circ\}} \qquad \mathsf{id} \frac{}{\Gamma\{a^\bullet, a^\circ\}}$$

$$\wedge^\bullet \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \frac{\Gamma\{A^\circ\} \qquad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}}$$

$$\vee^\bullet \frac{\Gamma\{A^\bullet, \Pi^\circ\} \qquad \Gamma\{B^\bullet, \Pi^\circ\}}{\Gamma\{A \vee B^\bullet, \Pi^\circ\}} \qquad \vee^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \quad \vee^\circ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset^\bullet \frac{\Gamma^{\Downarrow}\{A^\circ\} \qquad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}} \qquad \Box^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Box A^\circ\}}$$

$$\Diamond^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \mathsf{c} \frac{\Gamma\{\Delta^\bullet, \Delta^\bullet\}}{\Gamma\{\Delta^\bullet\}} \qquad \Diamond^\circ \frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

Figure 2: System NCK

- The $\bot^\bullet$-rule and the $\vee^\bullet$-rule have a restriction on where the output formula occurs in the context: it has to be in the same subtree of the sequent as the principal formula of the rule. The reason for this is the lack of k_3 (for the $\vee^\bullet$-rule) and k_5 (for the $\bot^\bullet$-rule).
- In the $\supset^\bullet$-rule (and also in the cut-rule described below), the 'output-pruning' is defined differently from [Str13]. There only the unique output formula is removed, whereas here the whole subtree containing the output formula is removed. The reason for this is the lack of the k_4-axiom.
- In [Str13] the structural rule $\mathsf{m}^{[]} \frac{\Gamma\{[\Delta_1], [\Delta_2]\}}{\Gamma\{[\Delta_1, \Delta_2]\}}$ is heavily used. This rule is not available to us here because it is not sound in the constructive setting. It corresponds to the k_4-axiom when the output formula occurs in Δ_1 or Δ_2.

Note that the id rule applies only to atomic formulas but, as usual with sequent style systems, the general form is derivable and this can be shown by a straightforward induction.

Proposition 3.1. *The rule* $\mathsf{id} \frac{}{\Gamma\{A^\bullet, A^\circ\}}$ *is derivable in* NCK.

In the course of this paper we make use of the following structural rules:

$$\mathsf{nec}^{[]} \frac{\Gamma}{[\Gamma]} \qquad \mathsf{w} \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta^\bullet\}} \qquad \mathsf{cut} \frac{\Gamma^{\Downarrow}\{A^\circ\} \qquad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}} \tag{3.1}$$

called *necessitation*, *weakening*, and *cut*, respectively. These rules are not part of the system, but we will later see that they are all admissible. Note that in the weakening rule Δ has to be an LHS sequent, as is the case for the contraction rule c, as one might expect in an intuitionistic setting. The cut rule makes use of the output pruning in the same way as the $\supset^\bullet$-rule.

We now turn to the rules for the axioms in (1.2). For d, t and 4, the corresponding rules are shown in Figure 3, and they coincide with those in [Str13].

$$\mathsf{d}^\circ \frac{\Gamma\{[A^\circ]\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{t}^\circ \frac{\Gamma\{A^\circ\}}{\Gamma\{\Diamond A^\circ\}} \qquad \mathsf{4}^\circ \frac{\Gamma\{[\Diamond A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}$$

$$\mathsf{d}^\bullet \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Box A^\bullet\}} \qquad \mathsf{t}^\bullet \frac{\Gamma\{A^\bullet\}}{\Gamma\{\Box A^\bullet\}} \qquad \mathsf{4}^\bullet \frac{\Gamma\{[\Box A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [\Delta]\}}$$

Figure 3: Constructive $\Diamond^\circ$- and $\Box^\bullet$-rules for the axioms d, t, and $\mathsf{4}$.

$$\mathsf{d}^{[]} \frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}} \qquad \mathsf{t}^{[]} \frac{\Gamma\{[\Sigma]\}}{\Gamma\{\Sigma\}} \qquad \mathsf{b}^{[]} \frac{\Gamma\{[[\Sigma], \Delta]\}}{\Gamma\{\Sigma, [\Delta]\}} \qquad \mathsf{4}^{[]} \frac{\Gamma\{[\Sigma]\}}{\Gamma\{[[\Sigma]]\}} \qquad \mathsf{5}^{[]} \frac{\Gamma\{[[\Sigma], \Delta]\}}{\Gamma\{[\Sigma], [\Delta]\}}$$

Figure 4: Structural rules for the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$

For the b and $\mathsf{5}$ axioms, the rules given in [Str13] (themselves adapted from the classical setting [Brü09]) are not sound in the constructive setting, again due to the lack of k_4. For b, one could restrict the rules of [Str13] in the following way,

$$\mathsf{b}^\circ_{int} \frac{\Gamma\{[\Delta], A^\circ\}}{\Gamma\{[\Delta, \Diamond A^\circ]\}} \quad \rightsquigarrow \quad \mathsf{b}^\circ_{con} \frac{\Gamma\{A^\circ\}}{\Gamma\{[\Delta^\bullet, \Diamond A^\circ]\}} \tag{3.2}$$

$$\mathsf{b}^\bullet_{int} \frac{\Gamma\{[\Delta], A^\bullet\}}{\Gamma\{[\Delta, \Box A^\bullet]\}} \quad \rightsquigarrow \quad \mathsf{b}^\bullet_{con} \frac{\Gamma\{A^\bullet\}}{\Gamma\{[\Delta^\bullet, \Box A^\bullet]\}} \tag{3.3}$$

in order to regain soundness. However such a system is not yet complete as, for example, the formula $\Diamond(\Box A \vee \bot) \supset A$ is no longer provable in the cut-free system.

To address this problem, we introduce structural rules in Figure 4 which were used during the cut-elimination proofs of [Brü09] and [Str13]. These rules are identical to the ones in [Brü09] and [Str13] for d, t, and b. For $\mathsf{4}$, our rule is slightly weaker than the one in [Str13], again due to the lack of k_4.

Finally, for $\mathsf{5}$, the situation is more subtle. Again, the general versions of the logical rules $\mathsf{5}^\bullet$ and $\mathsf{5}^\circ$ from [Str13] are no longer sound due to the lack of k_4. These $\mathsf{5}^\bullet$ and $\mathsf{5}^\circ$ rules can each be decomposed into three rules performing smaller inference steps, but unfortunately all three of these are unsound, and only the first can be made sound by incorporating weakening as shown for b° and $\mathsf{b}^\bullet$ in (3.2) and (3.3) above. As expected, the resulting system is again incomplete.

Perhaps surprisingly, the structural rule $\mathsf{5}^{[]}$ used in [Brü09] and [Str13] is also no longer sound in the constructive setting due to the lack of k_4. However, similar to the rules $\mathsf{5}^\bullet$ and $\mathsf{5}^\circ$ from [Str13], the rule $\mathsf{5}^{[]}$ of [Brü09] and [Str13] can be decomposed into three smaller rules and the first of them, shown in Figure 4, is sound. The resulting system, in the constructive setting, is now complete.

In the remainder of this paper, we use the following notation. For a set of axioms $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$ we write $\mathsf{X}^{[]}$ to denote the set of corresponding structural rules shown in Figure 4, and if $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$, we write $\mathsf{X}^{\circ\!\bullet}$ for the set of corresponing $\Box^\bullet$- and $\Diamond^\circ$-rules shown in Figure 3.

4. Soundness

To our knowledge there are no standard Kripke semantics for all the various constructive modal logics and consideration of this issue is beyond the scope of this work. Therefore we show soundness of our rules with respect to the Hilbert system.

For this we define HCK to be some complete set of axioms for intuitionistic propositional logic extended by the axioms k_1 and k_2, shown in (1.1), together with the rules mp for *modus ponens* and nec for *necessitation*:

$$\mathsf{mp}\,\frac{A \quad A \supset B}{B} \qquad\qquad \mathsf{nec}\,\frac{A}{\Box A} \tag{4.1}$$

Soundness is now stated in the following theorem:

Theorem 4.1. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$*, let* $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, and let* $\mathsf{r}\,\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma}$ *(for* $n \in \{0,1,2\}$*) be an instance of a rule in* $\mathsf{NCK} + \mathsf{X}^{\circ\!\!\bullet} + \mathsf{Y}^{[]}$*. Then:*

(i) the formula $fm(\Gamma_1) \wedge \cdots \wedge fm(\Gamma_n) \supset fm(\Gamma)$ *is provable in* $\mathsf{HCK} + \mathsf{X} + \mathsf{Y}$*, and*
(ii) whenever a sequent Γ *is provable in* $\mathsf{NCK} + \mathsf{X}^{\circ\!\!\bullet} + \mathsf{Y}^{[]}$*, then* $fm(\Gamma)$ *is provable in* $\mathsf{HCK} + \mathsf{X} + \mathsf{Y}$*.*

Clearly, (ii) follows immediately from (i) using an induction on the size of the derivation. For proving (i), we start with the axioms:

Lemma 4.2. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* $\Gamma\{\ \}$ *be an output context, and* Π° *be an RHS-sequent. Then* $fm(\Gamma\{a^\bullet, a^\circ\})$ *and* $fm(\Gamma\{\bot^\bullet, \Pi^\circ\})$ *are provable in* $\mathsf{HCK} + \mathsf{X}$*.*

Proof. By induction on the structure of $\Gamma\{\ \}$. □

For showing soundness of the inference rules with one premise, we first have to verify that the deep inference reasoning remains valid in the constructive setting.

Lemma 4.3. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, and let* A*,* B*, and* C *be formulas.*

(i) If $A \supset B$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $(C \supset A) \supset (C \supset B)$*.*
(ii) If $A \supset B$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $(B \supset C) \supset (A \supset C)$*.*
(iii) If $A \supset B$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $(C \wedge A) \supset (C \wedge B)$*.*
(iv) If $A \supset B$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $\Box A \supset \Box B$*.*
(v) If $A \supset B$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $\Diamond A \supset \Diamond B$*.*

Proof. (i), (ii) and (iii) follow by completeness of HCK over intuitionistic logic. (iv) and (v) follow by necessitation and k_1 or k_2, respectively. □

Lemma 4.4. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* Δ *and* Σ *be full sequents, and let* $\Gamma\{\ \}$ *be an output context. If* $fm(\Delta) \supset fm(\Sigma)$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $fm(\Gamma\{\Delta\}) \supset fm(\Gamma\{\Sigma\})$*.*

Proof. Induction on the structure of $\Gamma\{\ \}$ (see Observation 2.2), using Lemma 4.3.(i) and (iv). □

Lemma 4.5. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* Δ *and* Σ *be LHS-sequents, and* $\Gamma\{\ \}$ *an input context. If* $fm(\Sigma) \supset fm(\Delta)$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $fm(\Gamma\{\Delta\}) \supset fm(\Gamma\{\Sigma\})$*.*

Proof. As in [Str13]. By Observation 2.2, $\Gamma\{\ \} = \Gamma'\{\Lambda\{\ \}, \Pi\}$ for some $\Gamma'\{\ \}$ and $\Lambda\{\ \}$ and Π. By induction on $\Lambda\{\ \}$, using Lemma 4.3.(iii) and (v), we get $fm(\Lambda\{\Sigma\}) \supset fm(\Lambda\{\Delta\})$, and from Lemma 4.3.(ii) it then follows that $(fm(\Lambda\{\Delta\}) \supset fm(\Pi)) \supset (fm(\Lambda\{\Sigma\}) \supset fm(\Pi))$. Now the statement follows from Lemma 4.4. □

Lemma 4.6. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, and let* $\mathsf{r}\,\dfrac{\Gamma_1}{\Gamma_2}$ *be an instance of* w, c, $\vee^\circ$, $\Box^\circ$, $\Diamond^\circ$, $\supset^\circ$, $\wedge^\bullet$, $\Diamond^\bullet$*, or* $\Box^\bullet$*. Then* $\mathit{fm}(\Gamma_1) \supset \mathit{fm}(\Gamma_2)$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*.*

Proof. For the rules $\vee^\circ$, $\Box^\circ$, $\Diamond^\circ$, $\supset^\circ$ this follows immediately from Lemma 4.4, where for $\Diamond^\circ$ we need the k_2-axiom. For the other rules we apply Lemma 4.5. Note that for the $\Box^\bullet$-rule we need a case distinction: If the output formula occurs inside Δ, then we use k_1 and Lemma 4.4. If the output formula occurs inside the context $\Gamma\{\ \}$, then we use k_2 and Lemma 4.5. □

Let us now turn to showing the soundness of the branching rules $\wedge^\circ$, $\vee^\bullet$, $\supset^\bullet$, and cut. For this, we develop appropriate versions of Lemmas 4.3 and 4.4 that deal with branching behaviour. Note that contrary to the intuitionistic case in [Str13], we do not have such a version of Lemma 4.5 in the constructive setting. This is due to the lack of axiom k_3.

Lemma 4.7. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, and let* A*,* B*,* C*, and* D *be formulas.*

(i) If $A \wedge B \supset C$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $(D \supset A) \wedge (D \supset B) \supset (D \supset C)$*.*
(ii) If $A \wedge B \supset C$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $(D \supset A) \wedge (D \wedge C) \supset (D \wedge B)$*.*
(iii) If $A \wedge B \supset C$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $\Box A \wedge \Box B \supset \Box C$*.*
(iv) If $A \wedge B \supset C$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $\Box A \wedge \Diamond C \supset \Diamond B$*.*

Proof. (i) and (ii) follow by completeness of HCK over intuitionistic logic. (iii) and (iv) follow by necessitation, distributivity of $\Box$ over $\wedge$, and k_1 or k_2 respectively. □

Lemma 4.8. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, let* Δ_1*,* Δ_2*, and* Σ *be full sequents, and let* $\Gamma\{\ \}$ *be an output context. If* $\mathit{fm}(\Delta_1) \wedge \mathit{fm}(\Delta_2) \supset \mathit{fm}(\Sigma)$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*, then so is* $\mathit{fm}(\Gamma\{\Delta_1\}) \wedge \mathit{fm}(\Gamma\{\Delta_2\}) \supset \mathit{fm}(\Gamma\{\Sigma\})$*.*

Proof. Induction on the structure of $\Gamma\{\ \}$, using Lemma 4.7.(i) and (iii). □

Lemma 4.9. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$*, and let* $\mathsf{r}\,\dfrac{\Gamma_1 \quad \Gamma_2}{\Gamma_3}$ *be an instance of* $\wedge^\circ$*,* $\vee^\bullet$*,* $\supset^\bullet$*, or* cut*. Then* $\mathit{fm}(\Gamma_1) \wedge \mathit{fm}(\Gamma_2) \supset \mathit{fm}(\Gamma_3)$ *is provable in* $\mathsf{HCK} + \mathsf{X}$*.*

Proof. For the $\wedge^\circ$- and $\vee^\bullet$-rules, this follows immediately from Lemma 4.8 and provable formulas $A \wedge B \supset A \wedge B$ and $(A \supset C) \wedge (B \supset C) \supset (A \vee B) \supset C$, respectively. For $\supset^\bullet$, note that by Observation 2.2 and Definition 2.3, the rule is of shape

$$\supset^\bullet \frac{\Gamma'\{\Lambda\{A^\circ\}\} \qquad \Gamma'\{\Lambda\{B^\bullet\}, \Pi^\circ\}}{\Gamma'\{\Lambda\{A \supset B^\bullet\}, \Pi^\circ\}}$$

where $\Gamma'\{\ \}$, $\Lambda\{\ \}$, and $\Pi\{\ \}$ are output contexts. In particular, let

$$\Lambda\{\ \} = \Lambda_0, [\Lambda_1, [\ldots, [\Lambda_n, \{\ \}]\ldots]] \quad .$$

Now let $P = \mathit{fm}(\Pi^\circ)$ and $L_i = \mathit{fm}(\Lambda_i)$ for $i = 0 \ldots n$, and let

$$L_X = \mathit{fm}(\Lambda\{A^\circ\}) = L_0 \supset \Box(L_1 \supset \Box(L_2 \supset \Box(\cdots \supset \Box(L_n \supset A)\cdots)))$$
$$L_Y = \mathit{fm}(\Lambda\{B^\bullet\}) = L_0 \wedge \Diamond(L_1 \wedge \Diamond(L_2 \wedge \Diamond(\cdots \wedge \Diamond(L_n \wedge B)\cdots)))$$
$$L_Z = \mathit{fm}(\Lambda\{A \supset B^\bullet\}) = L_0 \wedge \Diamond(L_1 \wedge \Diamond(L_2 \wedge \Diamond(\cdots \wedge \Diamond(L_n \wedge (A \supset B))\cdots)))$$

For being able to apply Lemma 4.8, we need to show that $(L_X \wedge (L_Y \supset P)) \supset (L_Z \supset P)$ is provable in $\mathsf{HCK} + \mathsf{X}$. But this follows from $L_X \wedge L_Z \supset L_Y$, which can be shown provable in $\mathsf{HCK} + \mathsf{X}$ using an induction on n together with Lemma 4.7.(ii) and (iv). For the cut-rule we additionally observe that $A \supset A$ is always provable. □

This completes the proof of soundness for NCK. Let us now turn to the rules in Figures 3 and 4.

Lemma 4.10. *Let S and D be arbitrary formulas. Then we have the following:*

(i) $(S \wedge \Diamond D) \supset \Diamond(\Diamond S \wedge D)$ *is a theorem of* $\mathsf{HCK} + \mathsf{b}$.
(ii) $\Box(D \supset \Box S) \supset (\Diamond D \supset S)$ *is a theorem of* $\mathsf{HCK} + \mathsf{b}$.
(iii) $\Box(\Diamond S \supset D) \supset (S \supset \Box D)$ *is a theorem of* $\mathsf{HCK} + \mathsf{b}$.
(iv) $(\Diamond S \wedge \Diamond D) \supset \Diamond(\Diamond S \wedge D)$ *is a theorem of* $\mathsf{HCK} + \mathsf{5}$.
(v) $\Box(D \supset \Box S) \supset (\Diamond D \supset \Box S)$ *is a theorem of* $\mathsf{HCK} + \mathsf{5}$.
(vi) $\Box(\Diamond S \supset D) \supset (\Diamond S \supset \Box D)$ *is a theorem of* $\mathsf{HCK} + \mathsf{5}$.

Proof. We already have shown that all rules of $\mathsf{NCK} + \mathsf{cut}$ are sound with respect to HCK. Thus, it suffices to prove the formulas within $\mathsf{NCK} + \mathsf{cut} + \mathsf{b}$ and $\mathsf{NCK} + \mathsf{cut} + \mathsf{5}$, respectively.

(i)

$$
{\supset^\circ}\dfrac{{\wedge^\bullet}\dfrac{{\Diamond^\bullet}\dfrac{{\Diamond^\circ}\dfrac{{\wedge^\circ}\dfrac{\mathsf{w}\dfrac{\mathsf{cut}\dfrac{\mathsf{w}\dfrac{\mathsf{b}\dfrac{}{S \supset \Box\Diamond S^\circ}}{S^\bullet, S \supset \Box\Diamond S^\circ} \qquad {\supset^\bullet}\dfrac{\mathsf{id}\dfrac{}{S^\bullet, S^\circ} \quad {\Box^\bullet}\dfrac{\mathsf{id}\dfrac{}{S^\bullet, [\Diamond S^\bullet, \Diamond S^\circ]}}{\Box\Diamond S^\bullet, S^\bullet, [\Diamond S^\circ]}}{S \supset \Box\Diamond S^\bullet, S^\bullet, [\Diamond S^\circ]}}{S^\bullet, [\Diamond S^\circ]}}{S^\bullet, [D^\bullet, \Diamond S^\circ]} \qquad \mathsf{id}\dfrac{}{S^\bullet, [D^\bullet, D^\circ]}}{S^\bullet, [D^\bullet, \Diamond S \wedge D^\circ]}}{S^\bullet, [D^\bullet], \Diamond(\Diamond S \wedge D)^\circ}}{S^\bullet, \Diamond D^\bullet, \Diamond(\Diamond S \wedge D)^\circ}}{S \wedge \Diamond D^\bullet, \Diamond(\Diamond S \wedge D)^\circ}}{(S \wedge \Diamond D) \supset \Diamond(\Diamond S \wedge D)^\circ}
$$

(ii)

$$
{\supset^\circ}\dfrac{{\supset^\circ}\dfrac{{\Diamond^\bullet}\dfrac{{\Box^\bullet}\dfrac{{\supset^\bullet}\dfrac{\mathsf{id}\dfrac{}{[D^\bullet, D^\circ]} \qquad \mathsf{w}\dfrac{\mathsf{cut}\dfrac{\mathsf{w}\dfrac{\mathsf{b}\dfrac{}{\Diamond \Box S \supset S^\circ}}{[\Box S^\bullet], \Diamond \Box S \supset S^\circ} \qquad {\supset^\bullet}\dfrac{{\Diamond^\circ}\dfrac{\mathsf{id}\dfrac{}{[\Box S^\bullet, \Box S^\circ]}}{[\Box S^\bullet], \Diamond \Box S^\circ} \quad \mathsf{id}\dfrac{}{[\Box S^\bullet], S^\bullet, S^\circ}}{[\Box S^\bullet], \Diamond \Box S \supset S^\bullet, S^\circ}}{[\Box S^\bullet], S^\circ}}{[\Box S^\bullet, D^\bullet], S^\circ}}{[D \supset \Box S^\bullet, D^\bullet], S^\circ}}{\Box(D \supset \Box S)^\bullet, [D^\bullet], S^\circ}}{\Box(D \supset \Box S)^\bullet, \Diamond D^\bullet, S^\circ}}{\Box(D \supset \Box S)^\bullet, \Diamond D \supset S^\circ}}{\Box(D \supset \Box S) \supset (\Diamond D \supset S)^\circ}
$$

(iii)

$$
{\supset^\circ}\dfrac{{\supset^\circ}\dfrac{{\Box^\circ}\dfrac{{\Box^\bullet}\dfrac{{\supset^\bullet}\dfrac{\overline{\overline{S^\bullet, [\Diamond S^\circ]}} \qquad \mathsf{id}\dfrac{}{S^\bullet, [D^\bullet, D^\circ]}}{S^\bullet, [\Diamond S \supset D^\bullet, D^\circ]}}{\Box(\Diamond S \supset D)^\bullet, S^\bullet, [D^\circ]}}{\Box(\Diamond S \supset D)^\bullet, S^\bullet, \Box D^\circ}}{\Box\Diamond S \supset D^\bullet, S \supset \Box D^\circ}}{\Box(\Diamond S \supset D) \supset (S \supset \Box D)^\circ}
$$

The double-line denotes a finite number of derivation steps. The derivation is omitted since it is identical to a sub-derivation tree of the first case.

(iv)

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\Diamond S^\bullet, [\Diamond S^\circ]$}
\LeftLabel{w}
\UnaryInfC{$\Diamond S^\bullet, \Diamond D^\bullet, [\Diamond S^\circ]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$\Diamond S^\bullet, [D^\bullet, D^\circ]$}
\LeftLabel{$\Diamond^\bullet$}
\UnaryInfC{$\Diamond S^\bullet, \Diamond D^\bullet, [D^\circ]$}
\LeftLabel{$\wedge^\circ$}
\BinaryInfC{$\Diamond S^\bullet, \Diamond D^\bullet, [\Diamond S \wedge D^\circ]$}
\LeftLabel{$\Diamond^\circ$}
\UnaryInfC{$\Diamond S^\bullet, \Diamond D^\bullet, \Diamond(\Diamond S \wedge D)^\circ$}
\LeftLabel{$\wedge^\bullet$}
\UnaryInfC{$\Diamond S \wedge \Diamond D^\bullet, \Diamond(\Diamond S \wedge D)^\circ$}
\LeftLabel{$\supset^\circ$}
\UnaryInfC{$(\Diamond S \wedge \Diamond D) \supset \Diamond(\Diamond S \wedge D)^\circ$}
\end{prooftree}
$$

Identical to a sub-derivation of the first case except that here we use 5 instead of b.

(v)

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$[D^\bullet, D^\circ]$}
\AxiomC{}
\doubleLine
\UnaryInfC{$[\Box S^\bullet], \Box S^\circ$}
\LeftLabel{w}
\UnaryInfC{$[D^\bullet, \Box S^\bullet], \Box S^\circ$}
\LeftLabel{$\supset^\bullet$}
\BinaryInfC{$[D^\bullet, D \supset \Box S^\bullet], \Box S^\circ$}
\LeftLabel{$\Box^\bullet$}
\UnaryInfC{$\Box(D \supset \Box S)^\bullet, [D^\bullet], \Box S^\circ$}
\LeftLabel{$\Diamond^\bullet$}
\UnaryInfC{$\Box(D \supset \Box S)^\bullet, \Diamond D^\bullet, \Box S^\circ$}
\LeftLabel{$\supset^\circ$}
\doubleLine
\UnaryInfC{$\Box(D \supset \Box S) \supset (\Diamond D \supset \Box S)^\circ$}
\end{prooftree}
$$

Identical to a sub-derivation tree of the second case, except that we use 5 instead of b.

(vi)

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\Diamond S^\bullet, [\Diamond S^\circ]$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$\Diamond S^\bullet, [D^\bullet, D^\circ]$}
\LeftLabel{$\supset^\bullet$}
\BinaryInfC{$\Diamond S^\bullet, [\Diamond S \supset D^\bullet, D^\circ]$}
\LeftLabel{$\Box^\bullet$}
\UnaryInfC{$\Box(\Diamond S \supset D)^\bullet, \Diamond S^\bullet, [D^\circ]$}
\LeftLabel{$\Box^\circ$}
\UnaryInfC{$\Box(\Diamond S \supset D)^\bullet, \Diamond S^\bullet, \Box D^\circ$}
\LeftLabel{$\supset^\circ$}
\doubleLine
\UnaryInfC{$\Box(\Diamond S \supset D) \supset (\Diamond S \supset \Box D)^\circ$}
\end{prooftree}
$$

□

Lemma 4.11. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$*, let* $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$*, let* $\mathsf{x} \in \mathsf{X}$*, let* $\mathsf{y} \in \mathsf{Y}$*, and let* $\mathsf{r}\,\dfrac{\Gamma_1}{\Gamma_2}$ *be an instance of* x° *or* $\mathsf{x}^\bullet$ *or* $\mathsf{y}^{[]}$*. Then* $fm(\Gamma_1) \supset fm(\Gamma_2)$ *is provable in* $\mathsf{HCK} + \mathsf{X} + \mathsf{Y}$*.*

Proof. For $\mathsf{d}^\bullet$, d°, $\mathsf{t}^\bullet$, t°, $\mathsf{4}^\bullet$, $\mathsf{4}^\circ$, $\mathsf{t}^{[]}$, and $\mathsf{4}^{[]}$ this follows immediately from Lemmas 4.4 and 4.5, and the corresponding axioms, shown in (1.2). For $\mathsf{d}^{[]}$ this says $\top \supset \Diamond\top$, which follows from $\top \supset \Box\top$ and the d-axiom. For $\mathsf{b}^{[]}$, we have to make a case analysis on where the output formula is. If it is in $\Gamma\{\ \}$, soundness of the rule follows from Lemma 4.10.(i) and Lemma 4.5. If it is in Σ, we use Lemma 4.10.(ii) and Lemma 4.4, and if it is in Δ, we use Lemma 4.10.(iii) and Lemma 4.4. For the rule $\mathsf{5}^{[]}$ we proceed similarly, using Lemma 4.10.(iv)–(vi) instead. □

Now we can put everything together to prove Theorem 4.1.

Proof of Theorem 4.1. Point (i) is just Lemmas 4.2, 4.6, 4.9, and 4.11. Point (ii) follows immediately from (i) using induction on the size of the derivation. □

$$\supset^\circ\dfrac{\mathsf{d}^\circ\dfrac{\Box^\bullet\dfrac{\mathsf{id}\dfrac{}{[A^\bullet, A^\circ]}}{\Box A^\bullet, [A^\circ]}}{\Box A^\bullet, \Diamond A^\circ}}{\Box A \supset \Diamond A^\circ}
\qquad
\wedge^\circ\dfrac{\supset^\circ\dfrac{\mathsf{t}^\circ\dfrac{\mathsf{id}\dfrac{}{A^\bullet, A^\circ}}{A^\bullet, \Diamond A^\circ}}{A \supset \Diamond A^\circ}\quad \supset^\circ\dfrac{\mathsf{t}^\bullet\dfrac{\mathsf{id}\dfrac{}{A^\bullet, A^\circ}}{\Box A^\bullet, A^\circ}}{\Box A \supset A^\circ}}{(A \supset \Diamond A) \wedge (\Box A \supset A)^\circ}$$

$$\wedge^\circ\dfrac{\supset^\circ\dfrac{\Diamond^\bullet\dfrac{\mathsf{b}^{[]}\dfrac{\Box^\bullet\dfrac{\mathsf{id}\dfrac{}{[[A^\bullet, A^\circ]]}}{[\Box A^\bullet, [A^\circ]]}}{[\Box A^\bullet], A^\circ}}{\Diamond \Box A^\bullet, A^\circ}}{\Diamond \Box A \supset A^\circ}\quad \supset^\circ\dfrac{\Box^\circ\dfrac{\mathsf{b}^{[]}\dfrac{\Diamond^\circ\dfrac{\mathsf{id}\dfrac{}{[[A^\bullet, A^\circ]]}}{[[A^\bullet], \Diamond A^\circ]}}{A^\bullet, [\Diamond A^\circ]}}{A^\bullet, \Box\Diamond A^\circ}}{A \supset \Box\Diamond A}}{(\Diamond \Box A \supset A) \wedge (A \supset \Box\Diamond A)^\circ}$$

$$\wedge^\circ\dfrac{\supset^\circ\dfrac{\Box^\circ\dfrac{\mathsf{4}^\bullet\dfrac{\mathsf{id}\dfrac{}{[\Box A^\bullet, \Box A^\circ]}}{\Box A^\bullet, [\Box A^\circ]}}{\Box A^\bullet, \Box \Box A^\circ}}{\Box A \supset \Box \Box A^\circ}\quad \supset^\circ\dfrac{\Diamond^\bullet\dfrac{\mathsf{4}^\circ\dfrac{\mathsf{id}\dfrac{}{[\Diamond A^\bullet, \Diamond A^\circ]}}{[\Diamond A^\bullet], \Diamond A^\circ}}{\Diamond\Diamond A^\bullet, \Diamond A^\circ}}{\Diamond\Diamond A \supset \Diamond A^\circ}}{(\Box A \supset \Box \Box A) \wedge (\Diamond\Diamond A \supset \Diamond A)^\circ}$$

$$\wedge^\circ\dfrac{\supset^\circ\dfrac{\Diamond^\bullet\dfrac{\Box^\circ\dfrac{\mathsf{5}^{[]}\dfrac{\Box^\bullet\dfrac{\mathsf{id}\dfrac{}{[[A^\bullet, A^\circ]]}}{[\Box A^\bullet, [A^\circ]]}}{[\Box A^\bullet], [A^\circ]}}{[\Box A^\bullet], \Box A^\circ}}{\Diamond \Box A^\bullet, \Box A^\circ}}{\Diamond \Box A \supset \Box A^\circ}\quad \supset^\circ\dfrac{\Diamond^\bullet\dfrac{\Box^\circ\dfrac{\mathsf{5}^{[]}\dfrac{\Diamond^\circ\dfrac{\mathsf{id}\dfrac{}{[[A^\bullet, A^\circ]]}}{[[A^\bullet], \Diamond A^\circ]}}{[A^\bullet], [\Diamond A^\circ]}}{[A^\bullet], \Box\Diamond A^\circ}}{\Diamond A^\bullet, \Box\Diamond A^\circ}}{\Diamond A \supset \Box\Diamond A^\circ}}{(\Diamond \Box A \supset \Box A) \wedge (\Diamond A \supset \Box\Diamond A)^\circ}$$

Figure 5: Proofs of the axioms d, t, b, $\mathsf{4}$, and $\mathsf{5}$ in our system

After having established soundness of our system, we can use it to make some interesting observations. Suprisingly, the b-axiom entails the axioms k_3 and k_5 (shown in (1.1) in the introduction), as can be seen by the following two derivations in $\mathsf{NCK} + \mathsf{b}^{[]}$:

$$\supset^\circ\dfrac{\Diamond^\bullet\dfrac{\mathsf{b}^{[]}\dfrac{\vee^\bullet\dfrac{\vee^\circ\dfrac{\mathsf{b}^{[]}\dfrac{\Diamond^\circ\dfrac{\mathsf{id}\dfrac{}{[[[A^\bullet, A^\circ]]]}}{[[[A^\bullet], \Diamond A^\circ]]}}{[A^\bullet, [\Diamond A^\circ]]}}{[A^\bullet, [\Diamond A \vee \Diamond B^\circ]]}\quad \vee^\circ\dfrac{\mathsf{b}^{[]}\dfrac{\Diamond^\circ\dfrac{\mathsf{id}\dfrac{}{[[[B^\bullet, B^\circ]]]}}{[[[B^\bullet], \Diamond B^\circ]]}}{[B^\bullet, [\Diamond B^\circ]]}}{[B^\bullet, [\Diamond A \vee \Diamond B^\circ]]}}{[A \vee B^\bullet, [\Diamond A \vee \Diamond B^\circ]]}}{[A \vee B^\bullet], \Diamond A \vee \Diamond B^\circ}}{\Diamond(A \vee B)^\bullet, \Diamond A \vee \Diamond B^\circ}}{\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)^\circ}
\quad\text{and}\quad
\supset^\circ\dfrac{\Diamond^\bullet\dfrac{\mathsf{b}^{[]}\dfrac{\bot^\bullet\dfrac{}{[\bot^\bullet, [\bot^\circ]]}}{[\bot^\bullet], \bot^\circ}}{\Diamond\bot^\bullet, \bot^\circ}}{\Diamond\bot \supset \bot^\circ}
\tag{4.2}$$

Whereas for k_5 this can easily be shown directly in the Hilbert system, the proof of k_3 in $\mathsf{HCK} + \mathsf{b}$ is not so simple. From our cut elimination result in Section 6 it will follow that the $\mathsf{5}$ axiom alone is not enough to derive k_3 or k_5. But since b is derivable in $\mathsf{CS5}$, both k_3 or k_5 are derivable in $\mathsf{CS5}$.

5. Completeness

Completeness is also shown with respect to the Hilbert system. This is in fact very similar to the completess proof for intuitionistic modal logic given in [Str13]. To simplify our cut elimination argument in Section 6 we will put a restriction on the $\Diamond^\bullet$-rule: We define system NCK' to be NCK with the $\Diamond^\bullet$-rule replaced by

$$\hat{\Diamond}^\bullet\,\dfrac{\Gamma\{[A^\bullet], \Pi^\circ\}}{\Gamma\{\Diamond A^\bullet, \Pi^\circ\}} \tag{5.1}$$

Theorem 5.1. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$ *and* $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$. *Then every formula that is provable in* $\mathsf{HCK} + \mathsf{X} + \mathsf{Y}$ *is provable in* $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]} + \mathsf{cut}$.

Proof. Clearly, all axioms of propositional intuitionistic logic are provable in NCK'. The axioms k_1 and k_2 are provable in NCK', by the same derivations as in [Str13], so we do not repeat them here. Note that the derivations for k_3, k_4, and k_5 of [Str13] are not valid in our setting because of the restictions to the $\vee^\bullet$-, $\supset^\bullet$-, and $\bot^\bullet$-rules, respectively. Figure 5 shows that each axiom $\mathsf{x} \in \mathsf{X} \cup \mathsf{Y}$ is provable in $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$. Finally, the rules mp and nec, shown in (4.1), can be simulated by the rules cut and $\mathsf{nec}^{[]}$, shown in (3.1). Then, the $\mathsf{nec}^{[]}$-rule is admissible, which can be seen by a straightforward induction on the size of the derivation. □

In the next section we show cut elimination for $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$, yielding completeness for the cut-free system. However, this is not achieved for every subset of $\mathsf{X} \cup \mathsf{Y}$ with $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, 4\}$ and $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. In fact, it can be shown that, for example, $\mathsf{NCK}' + 4^{[]}$ is not complete. On the other hand, we have:

Theorem 5.2. *Let* $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, 4\}$ *and* $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{b}, 5\}$*, such that if* $\mathsf{t} \in \mathsf{X}$ *and* $5 \in \mathsf{Y}$ *then* $\mathsf{b} \in \mathsf{Y}$*, and if* $\mathsf{b} \in \mathsf{Y}$ *or* $5 \in \mathsf{Y}$ *then* $4 \in \mathsf{X}$*. Then every formula that is provable in* $\mathsf{HCK} + \mathsf{X} + \mathsf{Y}$ *is also provable in* $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$.

Looking back at the cube in Figure 1, we can see that Theorem 5.2 gives us cut-free systems for the logics CK, CK4, CK45, CD, CD4, CD45, CT, CS4, and CS5. The logics for which our cut elimination proof does not work are CKB, CK5, CKB5, CD5, CDB, and CTB.

6. Cut Elimination

By inspection of the statement of Theorem 5.2, we have that t and 4 must be present as logical rules, and b and 5 as structural rules, whereas d can be present in either variation. This is due to the following result, whose proof is straightforward.

Proposition 6.1. *(i) The rules* $\mathsf{d}^\bullet$ *and* d° *are derivable in* $\{\Diamond^\bullet, \mathsf{d}^{[]}\}$ *and* $\{\Box^\circ, \mathsf{d}^{[]}\}$*, respectively. (ii) The rule* $\mathsf{d}^{[]}$ *is admissible for any subsystem of* $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$*, provided* $\mathsf{d} \in \mathsf{X}$.

However, our cut elimination argument becomes slightly simpler if we work with $\mathsf{d}^{[]}$ instead of $\mathsf{d}^\bullet$ and d°. To summarise, the following definition fixes the axiom sets our cut elimination proof deals with.

Definition 6.2. Let $\mathsf{X}, \mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, 4, 5\}$. We call the pair $\langle \mathsf{X}, \mathsf{Y} \rangle$ *safe* if $\mathsf{X} \subseteq \{\mathsf{t}, 4\}$ and $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{b}, 5\}$, such that if $\mathsf{t} \in \mathsf{X}$ and $5 \in \mathsf{Y}$ then $\mathsf{b} \in \mathsf{Y}$, and if $\mathsf{b} \in \mathsf{Y}$ or $5 \in \mathsf{Y}$ then $4 \in \mathsf{X}$.

Since our cut elimination strategy might seem unorthodox, we first explain some of the problems we encountered. Consider the following derivation:

$$
\mathsf{cut}\,\frac{\vee^\bullet\,\dfrac{\Gamma\{\Theta\{C^\bullet, \Diamond A^\circ\}\} \quad \Gamma\{\Theta\{B^\bullet, \Diamond A^\circ\}\}}{\Gamma\{\Theta\{C \vee B^\bullet, \Diamond A^\circ\}\}} \qquad \Diamond^\bullet\,\dfrac{\Gamma\{\Theta\{C \vee B^\bullet, [A^\bullet]\}, \Pi^\circ\}}{\Gamma\{\Theta\{C \vee B^\bullet, \Diamond A^\bullet\}, \Pi^\circ\}}}{\Gamma\{\Theta\{C \vee B^\bullet\}, \Pi^\circ\}} \tag{6.1}
$$

We cannot permute the instance of $\vee^\bullet$ under the cut because in general it is not applicable in $\Gamma\{\Theta\{C \vee B^\bullet\}, \Pi^\circ\}$. On the other hand, we cannot reduce the rank of the cut along the main connective of the cut formula $\Diamond A$, since there is no invertible rule for $\Diamond A^\circ$, and different things might happen in the left two branches. Furthermore, we cannot just impose the same restriction that we impose on the $\vee^\bullet$ rule also on the cut rule, because then we would not be able to reduce the cut rank in the ordinary $\Diamond^\circ$-$\Diamond^\bullet$ cases. The situation in (6.1) is the

$$\mathsf{s4}^{\circ}\,\frac{\Gamma\{\Delta\{\Diamond A^{\circ}\}\}}{\Gamma\{\Diamond A^{\circ},\Delta\{\emptyset\}\}}\quad \mathsf{s4}^{\bullet}\,\frac{\Gamma\{\Delta\{\Box A^{\bullet}\}\}}{\Gamma\{\Box A^{\bullet},\Delta\{\emptyset\}\}}\quad \mathsf{s4}^{\circ}_{\Diamond}\,\frac{\Gamma\{[\Delta\{A^{\circ}\}]\}}{\Gamma\{\Diamond A^{\circ},[\Delta\{\emptyset\}]\}}\quad \mathsf{s4}^{\bullet}_{\Box}\,\frac{\Gamma\{[\Delta\{A^{\bullet}\}]\}}{\Gamma\{\Box A^{\bullet},[\Delta\{\emptyset\}]\}}$$

$$\mathsf{sb}^{[]}\,\frac{\Gamma\{\Delta\{[\![\Sigma]\!]^{n}\}\}}{\Gamma\{\Sigma,\Delta\{\emptyset\}\}}\quad \mathsf{s5}^{[]}\,\frac{\Gamma\{\Delta\{[\Sigma]\}\}}{\Gamma\{[\Sigma],\Delta\{\emptyset\}\}}\quad \mathsf{s5b}^{[]}\,\frac{\Gamma\{\Delta\{[\![\Sigma]\!]^{k}\}\}}{\Gamma\{\Sigma,\Delta\{\emptyset\}\}}\quad \mathsf{sb5}^{[]}\,\frac{\Gamma\{\Delta\{[\![\Sigma]\!]^{k}\}\}}{\Gamma\{[\Sigma],\Delta\{\emptyset\}\}}$$

Figure 6: Super rules for $\mathsf{4}$, b, and $\mathsf{5}$, where n is the depth of $\Delta\{\ \}$ and $1 \leq k \leq n$.

reason we work with the rule $\hat{\Diamond}^{\bullet}$ instead of $\Diamond^{\bullet}$. Note that imposing the same restriction on all logical rules would make other permutation cases difficult.

In what follows we will use the shorthand $[\![\Gamma]\!]^{n}$ to denote Γ with n pairs of brackets around it, i.e. $\overbrace{[\cdots[}^{n}\ \Gamma\ \overbrace{]\cdots]}^{n}$. Also, we define the *depth* of a context $\Gamma\{\ \}$ to be the number of bracket pairs its hole appears in the scope of, i.e., the depth of $\Delta_0, [\Delta_1, [\ldots, [\Delta_n, \{\ \}]\ldots]]$ is n.

We consider *super rule* variants of the rules $\mathsf{4}^{\bullet}$, $\mathsf{4}^{\circ}$, $\mathsf{b}^{[]}$, and $\mathsf{5}^{[]}$, shown in Figure 6, obtained from unboundedly many applications of the corresponding normal rules in a certain way. For a safe pair $\langle \mathsf{X}, \mathsf{Y}\rangle$ of axioms, we define

$$\mathsf{X}^{\circ\!\!\bullet}_{\mathsf{s}} = \begin{cases} \mathsf{X}^{\circ\!\!\bullet} & \text{if } \mathsf{4} \notin \mathsf{X} \\ (\mathsf{X}^{\circ\!\!\bullet} \setminus \{\mathsf{4}^{\circ}, \mathsf{4}^{\bullet}\}) \cup \{\mathsf{s4}^{\circ}, \mathsf{s4}^{\bullet}, \mathsf{s4}^{\circ}_{\Diamond}, \mathsf{s4}^{\bullet}_{\Box}\} & \text{if } \mathsf{4} \in \mathsf{X} \end{cases}$$

and

$$\mathsf{Y}^{[]}_{\mathsf{s}} = \begin{cases} \mathsf{Y}^{[]} & \text{if } \mathsf{b}, \mathsf{5} \notin \mathsf{Y} \\ (\mathsf{Y}^{[]} \setminus \{\mathsf{b}^{[]}\}) \cup \{\mathsf{sb}^{[]}\} & \text{if } \mathsf{b} \in \mathsf{Y}, \mathsf{5} \notin \mathsf{Y} \\ (\mathsf{Y}^{[]} \setminus \{\mathsf{5}^{[]}\}) \cup \{\mathsf{s5}^{[]}\} & \text{if } \mathsf{b} \notin \mathsf{Y}, \mathsf{5} \in \mathsf{Y} \\ (\mathsf{Y}^{[]} \setminus \{\mathsf{b}^{[]}, \mathsf{5}^{[]}\}) \cup \{\mathsf{s5b}^{[]}, \mathsf{sb5}^{[]}\} & \text{if } \mathsf{b}, \mathsf{5} \in \mathsf{Y} \end{cases}$$

We need these variants in order to obtain height-preserving admissibility of certain rules. We have the following proposition:

Proposition 6.3. *A sequent is provable in* $\mathsf{NCK}' + \mathsf{X}^{\circ\!\!\bullet}_{\mathsf{s}} + \mathsf{Y}^{[]}_{\mathsf{s}}$ *if and only if it is provable in* $\mathsf{NCK}' + \mathsf{X}^{\circ\!\!\bullet} + \mathsf{Y}^{[]}$.

Proof. One direction follows immediately from the observation that $\mathsf{4}^{\bullet}$ and $\mathsf{4}^{\circ}$ are special cases of $\mathsf{s4}^{\bullet}$ and $\mathsf{s4}^{\circ}$, respectively, and that $\mathsf{b}^{[]}$ is a special case of $\mathsf{sb}^{[]}$ and of $\mathsf{s5b}^{[]}$, and that $\mathsf{5}^{[]}$ is a special case of $\mathsf{s5}^{[]}$ and of $\mathsf{sb5}^{[]}$. Conversely, $\mathsf{s4}^{\bullet}$ and $\mathsf{s4}^{\circ}$ are just sequences of $\mathsf{4}^{\bullet}$ and $\mathsf{4}^{\circ}$, respectively, and $\mathsf{s4}^{\bullet}_{\Box}$ and $\mathsf{s4}^{\circ}_{\Diamond}$ are obtained by composing with $\Box^{\bullet}$ and $\Diamond^{\circ}$, respectively. Then $\mathsf{sb}^{[]}$ and $\mathsf{s5}^{[]}$ are just sequences of $\mathsf{b}^{[]}$ and $\mathsf{5}^{[]}$, respectively, whereas $\mathsf{s5b}^{[]}$ and $\mathsf{sb5}^{[]}$ use both $\mathsf{b}^{[]}$ and $\mathsf{5}^{[]}$. □

Our cut rule shown in (3.1) is not enough for our induction to work, when the rules $\mathsf{4}^{\bullet}$ and $\mathsf{4}^{\circ}$ are in the system. Thus, we addionally use the two rules

$$\Diamond\mathsf{cut}\,\frac{\Gamma^{\Downarrow}\{\Theta^{\bullet}\{\Diamond A^{\circ}\}\} \quad \Gamma\{\Diamond A^{\bullet}, \Theta^{\bullet}\{\emptyset\}\}}{\Gamma\{\Theta^{\bullet}\{\emptyset\}\}} \quad \text{and} \quad \Box\mathsf{cut}\,\frac{\Gamma^{\Downarrow}\{\Box A^{\circ}, (\Theta\{\emptyset\})^{\Downarrow}\} \quad \Gamma\{\Theta\{\Box A^{\bullet}\}\}}{\Gamma\{\Theta\{\emptyset\}\}}$$

which are just combinations of cut with towers of $\mathsf{4}^{\circ}$ and $\mathsf{4}^{\bullet}$, respectively. By Cut, we refer to the set $\{\mathsf{cut}, \Diamond\mathsf{cut}, \Box\mathsf{cut}\}$ or $\{\mathsf{cut}\}$, depending on whether $\mathsf{4}^{\bullet}$ and $\mathsf{4}^{\circ}$ are present or not, and we write $*\mathsf{cut}$ for any variant in Cut.

Throughout this section we fix the convention that, for any $*\mathsf{cut}$ step, the output cut formula occurs in the left premise, while the input cut formula occurs in the right premise.

Definition 6.4. For a formula A we define *depth*(A) inductively as follows:

$$\mathit{depth}(a) = \mathit{depth}(\bot) = 1 \qquad\qquad \mathit{depth}(\Box A) = \mathit{depth}(\Diamond A) = \mathit{depth}(A) + 1$$
$$\mathit{depth}(A \wedge B) = \mathit{depth}(A \vee B) = \mathit{depth}(A \supset B) = \max(\mathit{depth}(A), \mathit{depth}(B)) + 1$$

Given a cut step, as shown in (3.1), its *cut formula* is A, and its *rank* is *depth*(A).

Definition 6.5. The inference rules id, $\bot^\bullet$, $\wedge^\bullet$, $\vee^\bullet$, $\supset^\bullet$, $\Box^\bullet$, $\Diamond^\bullet$, $\mathsf{t}^\bullet$, and $\mathsf{s4}^\bullet_\Box$ are called *black destructing.* The *principal formula* of a black destructing rule instance is the input formula singled out in its conclusion in Figures 2, 3 and 6.

In other words a rule instance is black destructing if, considered bottom-up, it decomposes an input formula along its main connective, and that formula is its principal formula. In particular, note that $\mathsf{4}^\bullet$ and $\mathsf{s4}^\bullet$ are not black destructing.

Definition 6.6. An instance of cut is *anchored* if the rule immediately above it on the right is a black-destructing rule whose principal formula is the cut formula. We define the *value* of a cut-instance to be the pair $\langle r, s\rangle$, where r is its rank, and $s = 0$ if it is anchored and $s = 1$ if it is not anchored. We consider values to be lexicographically ordered. Finally, the *cut-value* of a derivation $\mathcal{D}$, denoted by $\boldsymbol{v}(\mathcal{D})$ is the multiset of the values of its cut-instances.

Our cut reduction now proceeds by an induction in the cut-value of a derivation, always considering a topmost cut. There are two main lemmas, one for reducing anchored cuts (Lemma 6.17), and one for reducing cuts that are not anchored (Lemma 6.12). For both of these lemmas we need, as usual, height preserving admissibility and invertibility of certain inference rules.

Definition 6.7. The *height* of a derivation $\mathcal{D}$, denoted by $h(\mathcal{D})$, is defined to be the length of the maximal branch in the derivation tree. We say that a rule r with one premise is *height preserving admissible* in a system S, if for each derivation $\mathcal{D}$ in S of r's premise there is a derivation $\mathcal{D}'$ of r's conclusion in S, such that $h(\mathcal{D}') \leq h(\mathcal{D})$. Similarly, a rule r is *height preserving invertible* in a system S, if for every derivation of the conclusion of r there are derivations for each of r's premises with at most the same height.

Proposition 6.8. *Let $\langle \mathsf{X}, \mathsf{Y}\rangle$ be a safe pair of axioms. Then all rules in $\mathsf{X}^{[]}$, as well as the rules w and nec are height preserving admissible for $\mathsf{NCK}' \cup \mathsf{X}_\mathsf{s}^{\LEFTcircle} \cup \mathsf{Y}_\mathsf{s}^{[]}$.*

Proof. For w and nec this is a straightforward induction on the height of the derivation. For $\mathsf{t}^{[]}$ and $\mathsf{4}^{[]}$ we permute steps upwards through the proof to show admissibility, preserving height of the other rules in each reduction. Notice that, for either step, any nontrivial overlap with a rule above must have a bracket in its conclusion. For $\mathsf{t}^{[]}$ we have the following nontrivial cases:

(1) $\mathsf{sb}^{[]} - \mathsf{t}^{[]}$. The only overlap possible is in the Δ part of a $\mathsf{sb}^{[]}$-step, so let $\Delta\{\ \} = \Delta_1\{[\Delta_2\{\ \}]\}$ with *depth*$(\Delta_2) = m$ and *depth*$(\Delta_1) = n$, and the permutation is as follows:

$$\mathsf{t}^{[]}\,\dfrac{\mathsf{sb}^{[]}\,\dfrac{\Gamma\{\Delta_1\{[\Delta_2\{[\![\Sigma]\!]^{m+n+1}\}]\}\}}{\Gamma\{\Sigma, \Delta_1\{[\Delta_2\{\emptyset\}]\}\}}}{\Gamma\{\Sigma, \Delta_1\{\Delta_2\{\emptyset\}\}\}} \quad\rightarrow\quad \mathsf{sb}^{[]}\,\dfrac{\mathsf{t}^{[]}\,\dfrac{\mathsf{t}^{[]}\,\dfrac{\Gamma\{\Delta_1\{[\Delta_2\{[\![\Sigma]\!]^{m+n+1}\}]\}\}}{\Gamma\{\Delta_1\{\Delta_2\{[\![\Sigma]\!]^{m+n+1}\}\}\}}}{\Gamma\{\Delta_1\{\Delta_2\{[\![\Sigma]\!]^{m+n}\}\}\}}}{\Gamma\{\Sigma, \Delta_1\{\Delta_2\{\emptyset\}\}\}}$$

and we can apply the induction hypothesis twice.

(2) $\mathsf{s4}^\circ_\Diamond - \mathsf{t}^{[]}$.

$$\mathsf{t}^{[]}\,\dfrac{\mathsf{s4}^\circ_\Diamond\,\dfrac{\Gamma\{[\Delta\{A^\circ\}]\}}{\Gamma\{\Diamond A^\circ, [\Delta\{\emptyset\}]\}}}{\Gamma\{\Diamond A^\circ, \Delta\{\emptyset\}\}} \quad\to\quad \star\,\dfrac{\mathsf{t}^{[]}\,\dfrac{\Gamma\{[\Delta\{A^\circ\}]\}}{\Gamma\{\Delta\{A^\circ\}\}}}{\Gamma\{\Diamond A^\circ, \Delta\{\emptyset\}\}}$$

where $\star$ is t° if the hole of $\Delta\{\ \}$ has depth 0 and $\mathsf{s4}^\circ_\Diamond$ otherwise.

(3) $\mathsf{s4}^\bullet_\Box - \mathsf{t}^{[]}$. Similar to case 2.

(4) $\Diamond^\circ - \mathsf{t}^{[]}$.

$$\mathsf{t}^{[]}\,\dfrac{\Diamond^\circ\,\dfrac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}}{\Gamma\{\Diamond A^\circ, \Delta\}} \quad\to\quad \mathsf{t}^\circ\,\dfrac{\mathsf{t}^{[]}\,\dfrac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{A^\circ, \Delta\}}}{\Gamma\{\Diamond A^\circ, \Delta\}}$$

(5) $\Box^\bullet - \mathsf{t}^{[]}$. Similar to case 4.

(6) $\mathsf{s5b}^{[]} - \mathsf{t}^{[]}$. Similar to case 1.

(7) $\mathsf{sb5}^{[]} - \mathsf{t}^{[]}$. One overlap case is similar to case 1, and the other is given below.

$$\mathsf{t}^{[]}\,\dfrac{\mathsf{sb5}^{[]}\,\dfrac{\Gamma\{\Delta\{[\![\Sigma]\!]^k\}\}}{\Gamma\{[\Sigma], \Delta\{\emptyset\}\}}}{\Gamma\{\Sigma, \Delta\{\emptyset\}\}} \quad\to\quad \mathsf{s5b}^{[]}\,\dfrac{\Gamma\{\Delta\{[\![\Sigma]\!]^k\}\}}{\Gamma\{\Sigma, \Delta\{\emptyset\}\}}$$

And for $\mathsf{4}^{[]}$ we have the following nontrivial cases:

(8) $\mathsf{sb}^{[]} - \mathsf{4}^{[]}$. The only overlap possible is in the Δ part of a $\mathsf{sb}^{[]}$-step, so let $\Delta\{\ \} = \Delta_1\{[\Delta_2\{\ \}, \Delta_3]\}$ with $\mathit{depth}(\Delta_2) = m$ and $\mathit{depth}(\Delta_1) = n$, and the permutation is as follows:

$$\mathsf{4}^{[]}\,\dfrac{\mathsf{sb}^{[]}\,\dfrac{\Gamma\{\Delta_1\{[\Delta_2\{[\![\Sigma]\!]^{m+n+1}\}]\}\}}{\Gamma\{\Sigma, \Delta_1\{[\Delta_2\{\emptyset\}]\}\}}}{\Gamma\{\Sigma, \Delta_1\{[[\Delta_2\{\emptyset\}]]\}\}} \quad\to\quad \mathsf{sb}^{[]}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\Gamma\{\Delta_1\{[\Delta_2\{[\![\Sigma]\!]^{m+n+1}\}]\}\}}{\Gamma\{\Delta_1\{[[\Delta_2\{[\![\Sigma]\!]^{m+n+1}\}]]\}\}}}{\Gamma\{\Delta_1\{[[\Delta_2\{[\![\Sigma]\!]^{m+n+2}\}]]\}\}}}{\Gamma\{\Sigma, \Delta_1\{[[\Delta_2\{\emptyset\}]]\}\}}$$

and we can apply the induction hypothesis twice.

(9) $\mathsf{s4}^\circ_\Diamond - \mathsf{4}^{[]}$.

$$\mathsf{4}^{[]}\,\dfrac{\mathsf{s4}^\circ_\Diamond\,\dfrac{\Gamma\{[\Delta\{A^\circ\}]\}}{\Gamma\{\Diamond A^\circ, [\Delta\{\emptyset\}]\}}}{\Gamma\{\Diamond A^\circ, [[\Delta\{\emptyset\}]]\}} \quad\to\quad \mathsf{s4}^\circ_\Diamond\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\Gamma\{[\Delta\{A^\circ\}]\}}{\Gamma\{[[\Delta\{A^\circ\}]]\}}}{\Gamma\{\Diamond A^\circ, [[\Delta\{\emptyset\}]]\}}$$

(10) $\mathsf{s4}^\bullet_\Box - \mathsf{4}^{[]}$. Similar to case 9.

(11) $\Diamond^\circ - \mathsf{4}^{[]}$.

$$\mathsf{4}^{[]}\,\dfrac{\Diamond^\circ\,\dfrac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [\Delta]\}}}{\Gamma\{\Diamond A^\circ, [[\Delta]]\}} \quad\to\quad \mathsf{s4}^\circ_\Diamond\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{[[A^\circ, \Delta]]\}}}{\Gamma\{\Diamond A^\circ, [[\Delta]]\}}$$

(12) $\Box^\bullet - \mathsf{4}^{[]}$. Similar to case 11.

(13) $\mathsf{t}^{[]} - \mathsf{4}^{[]}$.

$$\mathsf{4}^{[]}\,\dfrac{\mathsf{t}^{[]}\,\dfrac{\Gamma\{[\Sigma]\}}{\Gamma\{[[\Sigma]]\}}}{\Gamma\{[\Sigma]\}} \quad\to\quad \Gamma\{[\Sigma]\}$$

(14) $\mathsf{s5b}^{[]} - \mathsf{4}^{[]}$. Similar to case 8.

(15) $\mathsf{sb5}^{[]} - \mathsf{4}^{[]}$. One overlap case is similar to 8 and the other is given below.

$$
\mathsf{4}^{[]}\,\dfrac{\mathsf{sb5}^{[]}\,\dfrac{\Gamma\{\Delta\{[\![\Sigma]\!]^k\}\}}{\Gamma\{[\Sigma],\Delta\{\emptyset\}\}}}{\Gamma\{[[\Sigma]],\Delta\{\emptyset\}\}}
\quad\rightarrow\quad
\mathsf{sb5}^{[]}\,\dfrac{\mathsf{4}^{[]}\,\dfrac{\Gamma\{\Delta\{[\![\Sigma]\!]^k\}\}}{\Gamma\{\Delta\{[\![[\Sigma]]\!]^k\}\}}}{\Gamma\{[[\Sigma]],\Delta\{\emptyset\}\}}
$$

Note that also permutation over contraction does preserve hight since we can apply the induction hypothesis twice. □

Note that the variants $\mathsf{X}^{\LEFTcircle}_{\mathsf{s}}$ of $\mathsf{X}^{\LEFTcircle}$ and $\mathsf{Y}^{[]}_{\mathsf{s}}$ of $\mathsf{Y}^{[]}$ are needed to make Proposition 6.8 work. Without the "super-rules" we would not be able to preserve the height, and consequently would not be able to proceed by the induction hypothesis on the proof when eliminating $\mathsf{t}^{[]}$ and $\mathsf{4}^{[]}$ in the cases 1 and 8 above.

Proposition 6.9. *The rules $\wedge^\bullet$, $\vee^\bullet$, $\Diamond^\bullet$, $\wedge^\circ$, $\supset^\circ$, $\Box^\circ$, and $\supset^\bullet$ on the right premise, are height preserving invertible for $\mathsf{NCK}' \cup \mathsf{X}^{\LEFTcircle}_{\mathsf{s}} \cup \mathsf{Y}^{[]}_{\mathsf{s}}$.*

Proof. Straightforward induction on the height of the derivation. □

Before we can state our main lemmas, we need to define a restricted version of Buss' *logical flow-graphs* [Bus91].

Definition 6.10. We define the *(formula) flow-graph* of a derivation $\mathcal{D}$, denoted by $G(\mathcal{D})$ to be the directed graph whose vertices are all input formula occurrences in $\mathcal{D}$, and there is an edge between two formula occurrences if they are the same unaltered occurrence in premise and conclusion of an instance of an inference rule. This concerns all formula occurrences in $\Gamma\{\ \}$, Δ, Π, and Σ in the rules in Figures. 2, 3, 4, 6 and cut, as well as the occurrences of $\Box A^\bullet$ in the $\mathsf{4}^\bullet$ and $\mathsf{s4}^\bullet$ rules. The edges are always directed from premise to conclusion.

Let us emphasise that there are no edges between a formula occurrence and any of its subformulae that may occur in $G(\mathcal{D})$. For example, the principal $A \vee B^\bullet$ in the conclusion of an $\vee^\bullet$-rule is neither connected to the $A^\bullet$ nor the $B^\bullet$ in the premises. But every formula occurrence $\Gamma\{\{\ \},\Pi^\circ\}$ in the conclusion is connected via an edge to the same occurrence in each of the two premises. Thus, the flow-graph is essentially a set of trees, where branching occurs in the branching rules $\vee^\bullet$, $\supset^\bullet$, $\wedge^\circ$, and in a contraction because every formula occurrence in $\Delta^\bullet$ in the conclusion is connected to each of its copies in the premise.

Recall from Dfn. 6.5 the notion of a black-destructing rule, from Dfn. 6.6 the notion of an anchored cut, and our convention that an output cut formula is written on the left hand side of a $*\mathsf{cut}$ step and an input cut formula on the right.

Definition 6.11. A *cut path* in $G(\mathcal{D})$ is a maximal path that ends at the cut formula $A^\bullet$ in the right-hand side premise of a cut-instance. A cut path is *relevant* if it starts at the principal formula of a black destructing rule. Otherwise it is called *irrelevant.* A cut path is *left-free* if it never passes through a left-hand side premise of an instance of cut. A derivation $\mathcal{D}$ is *left-free* if all relevant cut paths in in $G(\mathcal{D})$ are left-free. An *origin* of $G(\mathcal{D})$ is the topmost vertex of a relevant cut path in $G(\mathcal{D})$. An origin is *anchored* if its cut path has length 0. A derivation is *anchored* if all its cuts are anchored.

To be clear, irrelevant cut-paths are exactly those that begin in the context of an axiom, i.e. in the $\Gamma\{\ \}$ part of a $\bot^\bullet$ or id step.

Notice that we are using the term 'anchored' to describe both cuts, as in Dfn. 6.6, and origins as in the definition above (as well as derivations). In particular we point out that, if all cuts are anchored, then they can only have one origin which is also anchored, and so all origins are anchored. Conversely, if all origins are anchored (which are only defined for relevant cut-paths), there may be some cuts that are not anchored in the derivation, namely those with only irrelevant cut-paths.

An anchored derivation, thus, is one all of whose cuts and origins are anchored, which is not the same as simply having all origins anchored. This subtlety is important in the proof of Lemma 6.12 below.

Lemma 6.12. *Let* $\langle \mathsf{X}, \mathsf{Y} \rangle$ *be a safe pair of axioms, and let* $\mathcal{D}$ *be a left-free derivation* $\mathcal{D}$ *in* $\mathsf{NCK}' + \mathsf{X}^{\LEFTcircle} + \mathsf{Y}^{[]} + \mathsf{Cut}$. *Then there is an anchored derivation* $\mathcal{D}'$ *in* $\mathsf{NCK}' + \mathsf{X}^{\LEFTcircle} + \mathsf{Y}^{[]} + \mathsf{Cut}$, *such that for each* $*\mathsf{cut}$ *in* $\mathcal{D}'$, *there is a* $*\mathsf{cut}$ *in* $\mathcal{D}$ *of the same rank.*

Before we can prove this lemma, we need some more technicalities.

Definition 6.13. An instance of cut in $\mathcal{D}$ is called *relevant* if it has at least one relevant cut path. Otherwise it is called *irrelevant*. The *relevant cut-value* of a derivation $\mathcal{D}$, denoted by $\boldsymbol{v}_{\mathbf{r}}(\mathcal{D})$, is the multiset of the values of its relevant cuts.

Lemma 6.14. *Let* $\langle \mathsf{X}, \mathsf{Y} \rangle$ *be a safe pair of axioms. Given a derivation* $\mathcal{D}$ *in* $\mathsf{NCK}' + \mathsf{X}^{\bullet} + \mathsf{X}^{\circ} + \mathsf{Y}^{[]} + \mathsf{Cut}$, *there is a derivation* $\mathcal{D}'$ *in* $\mathsf{NCK}' + \mathsf{X}^{\bullet} + \mathsf{X}^{\circ} + \mathsf{Y}^{[]} + \mathsf{Cut}$ *of the same conclusion, such that* $\mathcal{D}'$ *has no irrelevant cuts, and such that* $\boldsymbol{v}_{\mathbf{r}}(\mathcal{D}') \leq \boldsymbol{v}_{\mathbf{r}}(\mathcal{D})$.

Proof. We proceed by induction on the number of irrelevant cuts in $\mathcal{D}$. Consider the topmost one. We can replace

$$\mathsf{cut}\ \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^{\Downarrow}\{A^\circ\}\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma\{A^\bullet\}\end{array}}{\Gamma\{\emptyset\}} \qquad \text{by} \qquad \begin{array}{c}\mathcal{D}_2'\\ \Gamma\{\emptyset\}\end{array}$$

where $\mathcal{D}_2'$ is obtained from $\mathcal{D}_2$ by removing the $A^\bullet$ occurrence everywhere; this results in a correct derivation since, by irrelevance, $A^\bullet$ must occur in the context of an axiom. For $\Diamond\mathsf{cut}$ and $\Box\mathsf{cut}$ we proceed similarly. ∎

Fact 6.15. The rule $\Diamond\mathsf{cut}$ is derivable in $\{\mathsf{cut}, \mathsf{s4}^\circ\}$, and the rule $\Box\mathsf{cut}$ is derivable in $\{\mathsf{cut}, \mathsf{s4}^\bullet\}$.

Lemma 6.16. *Let* $\langle \mathsf{X}, \mathsf{Y} \rangle$ *be a safe pair of axioms. If* $4 \in \mathsf{X}$, *then the rules* $\mathsf{s4}^\bullet$ *and* $\mathsf{s4}^\circ$ *permute over any* $\mathsf{r} \in \mathsf{Y}^{[]}_{\mathsf{s}}$.

Proof. We show here how $\mathsf{s4}^\circ$ permutes over $\mathsf{sb}^{[]}$. There are two nontrivial interactions:

$$\mathsf{s4}^\circ\,\frac{\mathsf{sb}^{[]}\,\dfrac{\Gamma\{\Delta\{[\![\Sigma\{\Diamond A^\circ\}]\!]^n\}\}}{\Gamma\{\Sigma\{\Diamond A^\circ\}, \Delta\{\emptyset\}\}}}{\Gamma\{\Diamond A^\circ, \Sigma\{\emptyset\}, \Delta\{\emptyset\}\}} \quad \to \quad \mathsf{sb}^{[]}\,\frac{\mathsf{s4}^\circ\,\dfrac{\Gamma\{\Delta\{[\![\Sigma\{\Diamond A^\circ\}]\!]^n\}\}}{\Gamma\{\Diamond A^\circ, \Delta\{[\![\Sigma\{\emptyset\}]\!]^n\}\}}}{\Gamma\{\Diamond A^\circ, \Sigma\{\emptyset\}, \Delta\{\emptyset\}\}}$$

and

$$\mathsf{s4}^\circ\,\frac{\mathsf{sb}^{[]}\,\dfrac{\Gamma\{\Delta'\{[\![\Sigma]\!]^n\}\}}{\Gamma\{\Sigma, \Delta'\{\emptyset\}\}}}{\Gamma\{\Diamond A^\circ, \Sigma, \Delta\{\emptyset\}\}} \quad \to \quad \mathsf{sb}^{[]}\,\frac{\mathsf{s4}^\circ\,\dfrac{\Gamma\{\Delta'\{[\![\Sigma]\!]^n\}\}}{\Gamma\{\Diamond A^\circ, \Delta\{[\![\Sigma]\!]^n\}\}}}{\Gamma\{\Diamond A^\circ, \Sigma, \Delta\{\emptyset\}\}}$$

The other cases are similar. ∎

Proof of Lemma 6.12. We proceed by induction on the number of origins in $G(\mathcal{D})$ that are not anchored. If all origins are anchored, then we remove all irrelevant cuts using Lemma 6.14 and we are done. Otherwise, we pick a topmost origin that is not anchored. Now we proceed by an inner induction on the length of its cut path to show that there is a derivation in which the number of non-anchored origins has decreased. Consider the $*\mathsf{cut}$-instance connected to our origin and make a case analysis on the rule instance r on the right above it.

(1) If r is one of $\wedge^\bullet$, $\Diamond^\bullet$, $\supset^\circ$, $\Box^\circ$, we can reduce as follows:

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}} \qquad \mathsf{r}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\mathsf{Inv}_1)}{\rightsquigarrow} \quad \mathsf{r}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{Inv}_\mathsf{r}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}}}{\Gamma_1^{\Downarrow}\{A^\circ\}} \qquad \overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma_1\{\emptyset\}}}{\Gamma\{\emptyset\}} \tag{6.2}$$

where the Inv_r is eliminated by Proposition 6.9.

(2) If r is one of $\vee^\bullet$, $\wedge^\circ$, we can reduce as follows:

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}} \qquad \mathsf{r}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}} \quad \overset{\mathcal{D}_3}{\Gamma_2\{A^\bullet\}}}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\mathsf{r})}{\rightsquigarrow} \quad \mathsf{r}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{Inv}_\mathsf{r}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}}}{\Gamma_1^{\Downarrow}\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma_1\{\emptyset\}} \qquad \mathsf{cut}\,\dfrac{\mathsf{Inv}_\mathsf{r}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}}}{\Gamma_2^{\Downarrow}\{A^\circ\}} \quad \overset{\mathcal{D}_3}{\Gamma_2\{A^\bullet\}}}{\Gamma_2\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

where the Inv_r steps are eliminated by Proposition 6.9. Note that it can happen that the Inv_r-step is vacuous in the above because depending on the position of the output formula in $\Gamma\{\ \}$ it is possible that $\Gamma_1^{\Downarrow}\{\ \} = \Gamma^{\Downarrow}\{\ \}$ and $\Gamma_2^{\Downarrow}\{\ \} = \Gamma^{\Downarrow}\{\ \}$.

(3) If r is $\supset^\bullet$, there are two cases. The first is

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Delta^\bullet\{A^\circ\}\}} \qquad \supset^\bullet\,\dfrac{\overset{\mathcal{D}_2}{\Gamma^{\Downarrow}\{\Theta^{\Downarrow}\{B^\circ\}, \Delta^\bullet\{A^\bullet\}\}} \quad \overset{\mathcal{D}_3}{\Gamma\{\Theta\{C^\bullet\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{\Theta\{B \supset C^\bullet\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{\Theta\{B \supset C^\bullet\}, \Delta^\bullet\{\emptyset\}\}} \quad \overset{(\supset_2^\bullet)}{\rightsquigarrow}$$

$$\mathsf{c}\,\dfrac{\supset^\bullet\,\dfrac{\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Delta^\bullet\{A^\circ\}\}} \quad \mathsf{w}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma^{\Downarrow}\{\Theta^{\Downarrow}\{B^\circ\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma^{\Downarrow}\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Theta^{\Downarrow}\{B^\circ\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma^{\Downarrow}\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Theta^{\Downarrow}\{B^\circ\}, \Delta^\bullet\{\emptyset\}\}} \quad \mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Delta^\bullet\{A^\circ\}\}}}{\Gamma^{\Downarrow}\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, (\Theta\{C^\bullet\})^{\Downarrow}, \Delta^\bullet\{A^\circ\}\}} \quad \mathsf{w}\,\dfrac{\overset{\mathcal{D}_3}{\Gamma\{\Theta\{C^\bullet\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Theta\{C^\bullet\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Theta\{C^\bullet\}, \Delta^\bullet\{\emptyset\}\}}}{\Gamma\{(\Theta\{B \supset C^\bullet\})^{\Downarrow}, \Theta\{B \supset C^\bullet\}, \Delta^\bullet\{\emptyset\}\}}}{\Gamma\{\Theta\{B \supset C^\bullet\}, \Delta^\bullet\{\emptyset\}\}}$$

where the w steps are removed by Proposition 6.8. Left-freeness is preserved since the derivations initially on the right of the cut, $\mathcal{D}_2$ and $\mathcal{D}_3$, remain on the right of all cuts after the transformation. Finally, both of the new cuts have same rank as the initial cut, satisfying the requirement in the statement of the lemma.

Note that this case shows that we need an explicit contraction rule. Making contraction implicit in the $\supset^\bullet$-rule (as done in [Str13]) would not be enough, since we also need to duplicate the context $\Theta\{\ \}$.

In the case shown above, the output formula in the conclusion can be in $\Gamma\{\ \}$ or $\Theta\{\ \}$. There is another such case for $\supset^\bullet$ on the right branch, where the output formula in the conclusion is in $\Delta\{\ \}$. That case is simpler, and no extra contraction is needed.

(4) If r is one of $\Box^\bullet$, $\Diamond^\circ$, $\mathsf{t}^\bullet$, t°, c, or one of the $\mathsf{s4}$-rules, working entirely in the context of the cut formula $A^\bullet$, then there are contexts Γ', Γ'_1 such that we can reduce as follows:

$$
\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}} \qquad \mathsf{r}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}}
\quad \overset{(\mathsf{r})}{\rightsquigarrow} \quad
\mathsf{c}\,\dfrac{\mathsf{r}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{A^\circ\}}}{\Gamma_1'^{\Downarrow}\{A^\circ\}} \qquad \mathsf{w}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma_1\{A^\bullet\}}}{\Gamma'_1\{A^\bullet\}}}{\Gamma'_1\{\emptyset\}}}{\Gamma'\{\emptyset\}}}{\Gamma\{\emptyset\}}
\tag{6.3}
$$

where the instance of c duplicates as much material as necessary for performing r without losing the information about $\Gamma\{\ \}$. Then the w steps are removed by Proposition 6.8.

(5) If r is a $\mathsf{s4}^\bullet$ step moving the cut formula (which is of shape $\Box A^\bullet$), then we can inductively apply,

$$
\Box\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Box A^\circ, (\Theta\{\Delta\{\emptyset\}\})^{\Downarrow}\}} \qquad \mathsf{4}^\bullet\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{\Theta\{\Delta\{\Box A^\bullet\}\}\}}}{\Gamma\{\Theta\{\Box A^\bullet, \Delta\{\emptyset\}\}\}}}{\Gamma\{\Theta\{\Delta\{\emptyset\}\}\}}
\quad \overset{(\Box 4)}{\rightsquigarrow} \quad
\Box\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Box A^\circ, (\Theta\{\Delta\{\emptyset\}\})^{\Downarrow}\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Theta\{\Delta\{\Box A^\bullet\}\}\}}}{\Gamma\{\Theta\{[\Delta]\}\}}
$$

by decomposing the instance of $\mathsf{s4}^\bullet$ into several $\mathsf{4}^\bullet$ steps.

(6) If r is a c duplicating the cut formula, we can reduce as follows:

$$
\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Delta^\bullet\{A^\circ\}\}} \qquad \mathsf{c}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{\Delta^\bullet\{A^\bullet\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{\Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{\Delta^\bullet\{\emptyset\}\}}
\quad \overset{(\mathsf{c})}{\rightsquigarrow}
$$

$$
\mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Delta^\bullet\{A^\circ\}\}}}{\Gamma^{\Downarrow}\{\Delta^\bullet\{A^\circ\}, \Delta^\bullet\{\emptyset\}\}} \qquad \mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Delta^\bullet\{A^\circ\}\}}}{\Gamma^{\Downarrow}\{\Delta^\bullet\{A^\bullet\}, \Delta^\bullet\{A^\circ\}\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Delta^\bullet\{A^\bullet\}, \Delta^\bullet\{A^\bullet\}\}}}{\Gamma\{\Delta^\bullet\{A^\bullet\}, \Delta^\bullet\{\emptyset\}\}}}{\Gamma\{\Delta^\bullet\{\emptyset\}, \Delta^\bullet\{\emptyset\}\}}}{\Gamma\{\Delta^\bullet\{\emptyset\}\}}
\tag{6.4}
$$

Note that the number of origins is not increased because the derivation is left-free. Furthermore, we can choose the order of the two new cuts such that the origin we are working on belongs to the topmost cut. Thus, we can proceed by the induction hypothesis.

(7) If r is a $\mathsf{sb}^{[]}$ or $\mathsf{s5b}^{[]}$, such that the cut-formula $A^\bullet$ is inside Σ. Then there are two subcases.

(a) If the depth of $\Gamma\{\;\}$ is 0, then $\Gamma\{\Sigma, \Delta\{\emptyset\}\} = \Sigma, \Delta'\{\emptyset\}$ for some $\Delta'\{\;\}$. Thus, without loss of generality, we have:

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}} \qquad \mathsf{sb}^{[]}\,\dfrac{\overset{\mathcal{D}_2}{\Delta\{[\![\Sigma\{A^\bullet\}]\!]^n\}}}{\Sigma\{A^\bullet\}, \Delta\{\emptyset\}}}{\Sigma\{\emptyset\}, \Delta\{\emptyset\}} \quad \overset{(\mathsf{sb}^{[]})}{\rightsquigarrow}$$

$$\mathsf{c}\,\dfrac{\mathsf{sb}^{[]}\,\dfrac{\mathsf{cut}\,\dfrac{(n{+}1){*}\mathsf{w}\,\dfrac{2n{*}\mathsf{nec}\,\dfrac{\overset{\mathcal{D}_1}{\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}}}{[\![\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^{2n}}}{\Delta^{\Downarrow}\{[\![\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^n\}} \qquad \mathsf{w}\,\dfrac{\overset{\mathcal{D}_2}{\Delta\{[\![\Sigma\{A^\bullet\}]\!]^n\}}}{\Delta\{[\![\Sigma\{A^\bullet\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^n\}}}{\Delta\{[\![\Sigma\{\emptyset\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^n\}}}{\Sigma\{\emptyset\}, (\Delta\{\emptyset\})^{\Downarrow}, \Delta\{\emptyset\}}}{\Sigma\{\emptyset\}, \Delta\{\emptyset\}}$$

where we use Proposition 6.8 to remove the w- and nec-steps.

(b) If the depth of $\Gamma\{\;\}$ is ≥ 1, then $\Gamma\{\Sigma, \Delta\{\emptyset\}\} = \Gamma'\{[\Sigma, \Delta'\{\emptyset\}]\}$ for some $\Gamma'\{\;\}$ and $\Delta'\{\;\}$. Thus, without loss of generality, we have:

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{[\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}]\}} \qquad \mathsf{sb}^{[]}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[\Delta\{[\![\Sigma\{A^\bullet\}]\!]^n\}]\}}}{\Gamma\{[\Sigma\{A^\bullet\}, \Delta\{\emptyset\}]\}}}{\Gamma\{[\Sigma\{\emptyset\}, \Delta\{\emptyset\}]\}} \quad \overset{(\mathsf{sb}^{[]})}{\rightsquigarrow}$$

$$\mathsf{c}\,\dfrac{\mathsf{sb}^{[]}\,\dfrac{\mathsf{cut}\,\dfrac{(n{+}1){*}\mathsf{w}\,\dfrac{2n{*}\mathsf{4}^{[]}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{[\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}]\}}}{\Gamma^{\Downarrow}\{[[\![\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^{2n}]\}}}{\Gamma^{\Downarrow}\{[\Delta^{\Downarrow}\{[\![\Sigma^{\Downarrow}\{A^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^n\}]\}} \qquad \mathsf{w}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{[\Delta\{[\![\Sigma\{A^\bullet\}]\!]^n\}]\}}}{\Gamma\{[\Delta\{[\![\Sigma\{A^\bullet\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^n\}]\}}}{\Gamma\{[\Delta\{[\![\Sigma\{\emptyset\}, (\Delta\{\emptyset\})^{\Downarrow}]\!]^n\}]\}}}{\Gamma\{[\Sigma\{\emptyset\}, (\Delta\{\emptyset\})^{\Downarrow}, \Delta\{\emptyset\}]\}}}{\Gamma\{[\Sigma\{\emptyset\}, \Delta\{\emptyset\}]\}} \tag{6.5}$$

where we use Proposition 6.8 to remove the w- and $\mathsf{4}^{[]}$-steps. This case is the reason why we need the presence of 4 when we have b in our logic.

(8) If r is a $\mathsf{sb}^{[]}$ or $\mathsf{s5b}^{[]}$, such that the cut-formula $A^\bullet$ is inside $\Delta\{\emptyset\}$, we have that $\Delta\{\emptyset\} = \Delta_1\{\Delta_2\{\emptyset\}, \Delta_3\{\emptyset\}\}$ and we can reduce as follows:

$$\mathsf{cut}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^\Downarrow\{\Sigma^\Downarrow, \Delta_1^\Downarrow\{\Delta_2^\Downarrow\{A^\circ\}, (\Delta_3\{\emptyset\})^\Downarrow\}\}\end{array} \qquad \mathsf{sb}^{[]}\,\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Delta_1\{\Delta_2\{A^\bullet\}, \Delta_3\{[\![\Sigma]\!]^n\}\}\}\end{array}}{\Gamma\{\Sigma, \Delta_1\{\Delta_2\{A^\bullet\}, \Delta_3\{\emptyset\}\}\}}}{\Gamma\{\Sigma, \Delta_1\{\Delta_2\{\emptyset\}, \Delta_3\{\emptyset\}\}\}} \quad \overset{(\mathsf{sb}^{[]})}{\leadsto}$$

$$\mathsf{c}\,\dfrac{\mathsf{sb}^{[]}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^\Downarrow\{\Sigma^\Downarrow, \Delta_1^\Downarrow\{\Delta_2^\Downarrow\{A^\circ\}, (\Delta_3\{\emptyset\})^\Downarrow\}\}\end{array}}{\Gamma^\Downarrow\{\Sigma^\Downarrow, \Delta_1^\Downarrow\{\Delta_2^\Downarrow\{A^\circ\}, (\Delta_3\{[\![\Sigma]\!]^n\})^\Downarrow\}\}} \qquad \mathsf{w}\,\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Delta_1\{\Delta_2\{A^\bullet\}, \Delta_3\{[\![\Sigma]\!]^n\}\}\}\end{array}}{\Gamma\{\Sigma^\Downarrow, \Delta_1\{\Delta_2\{A^\bullet\}, \Delta_3\{[\![\Sigma]\!]^n\}\}\}}}{\Gamma\{\Sigma^\Downarrow, \Delta_1\{\Delta_2\{\emptyset\}, \Delta_3\{[\![\Sigma]\!]^n\}\}\}}}{\Gamma\{\Sigma^\Downarrow, \Sigma, \Delta_1\{\Delta_2\{\emptyset\}, \Delta_3\{\emptyset\}\}\}}}{\Gamma\{\Sigma, \Delta_1\{\Delta_2\{\emptyset\}, \Delta_3\{\emptyset\}\}\}} \tag{6.6}$$

where the w on the left is not be needed if $(\Delta_3\{\emptyset\})^\Downarrow = (\Delta_3\{[\![\Sigma]\!]^n\})^\Downarrow$. Note that this case can be seen as a special case of case 4 above.

(9) If r is a $\mathsf{s5}^{[]}$ or $\mathsf{sb5}^{[]}$, such that the cut-formula $A^\bullet$ is inside Σ, then the situation is similar to case 7b above:

$$\mathsf{cut}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^\Downarrow\{[\Sigma^\Downarrow\{A^\circ\}], (\Delta\{\emptyset\})^\Downarrow\}\end{array} \qquad \mathsf{s5}^{[]}\,\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Delta\{[\Sigma\{A^\bullet\}]\}\}\end{array}}{\Gamma\{[\Sigma\{A^\bullet\}], \Delta\{\emptyset\}\}}}{\Gamma\{[\Sigma\{\emptyset\}], \Delta\{\emptyset\}\}} \quad \overset{(\mathsf{s5}^{[]})}{\leadsto}$$

$$\mathsf{c}\,\dfrac{\mathsf{s5}^{[]}\,\dfrac{\mathsf{cut}\,\dfrac{(n{+}1){*}\mathsf{w}\,\dfrac{n{*}\mathsf{4}^{[]}\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma^\Downarrow\{[\Sigma^\Downarrow\{A^\circ\}, (\Delta\{\emptyset\})^\Downarrow]\}\end{array}}{\Gamma^\Downarrow\{[\![[\Sigma^\Downarrow\{A^\circ\}]]\!]^n, (\Delta\{\emptyset\})^\Downarrow\}}}{\Gamma^\Downarrow\{\Delta^\Downarrow\{[\Sigma^\Downarrow\{A^\circ\}]\}, (\Delta\{\emptyset\})^\Downarrow\}} \qquad \mathsf{w}\,\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma\{\Delta\{[\Sigma\{A^\bullet\}]\}\}\end{array}}{\Gamma\{\Delta\{[\Sigma\{A^\bullet\}]\}, (\Delta\{\emptyset\})^\Downarrow\}}}{\Gamma\{\Delta\{[\Sigma\{\emptyset\}]\}, (\Delta\{\emptyset\})^\Downarrow\}}}{\Gamma\{[\Sigma\{\emptyset\}], \Delta\{\emptyset\}, (\Delta\{\emptyset\})^\Downarrow\}}}{\Gamma\{[\Sigma\{\emptyset\}], \Delta\{\emptyset\}\}} \tag{6.7}$$

where we use Proposition 6.8 to remove the w- and $\mathsf{4}^{[]}$-steps. This case is the reason why we need the presence of $\mathsf{4}$ when we have $\mathsf{5}$ in our logic.

(10) If r is a $\mathsf{s5}^{[]}$ or $\mathsf{sb5}^{[]}$, such that the cut-formula $A^\bullet$ is inside $\Delta\{\emptyset\}$, then the situation is similar to case 8 above.

(11) If r is a $\mathsf{sb}^{[]}$, $\mathsf{s5b}^{[]}$, $\mathsf{s5}^{[]}$, or $\mathsf{sb5}^{[]}$, such that the cut-formula $A^\bullet$ is inside $\Gamma\{\ \}$, then we proceed as in case 4 above.

(12) Finally, if r is another cut, we can reduce as follows:

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Sigma^{\Downarrow}\{A^\circ\},(\Delta\{\emptyset\})^{\Downarrow}\}} \qquad \mathsf{cut}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma^{\Downarrow}\{(\Sigma\{A^\bullet\})^{\Downarrow},\Delta^{\Downarrow}\{B^\circ\}\}} \quad \overset{\mathcal{D}_3}{\Gamma\{\Sigma\{A^\bullet\},\Delta\{B^\bullet\}\}}}{\Gamma\{\Sigma\{A^\bullet\},\Delta\{\emptyset\}\}}}{\Gamma\{\Sigma\{\emptyset\},\Delta\{\emptyset\}\}} \quad \overset{(\mathsf{cut})}{\leadsto}$$

$$\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_2'}{\Gamma^{\Downarrow}\{(\Sigma\{\emptyset\})^{\Downarrow},\Delta^{\Downarrow}\{B^\circ\}\}} \qquad \mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma^{\Downarrow}\{\Sigma^{\Downarrow}\{A^\circ\},(\Delta\{\emptyset\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{\Sigma^{\Downarrow}\{A^\circ\},(\Delta\{B^\bullet\})^{\Downarrow}\}} \quad \overset{\mathcal{D}_3}{\Gamma\{\Sigma\{A^\bullet\},\Delta\{B^\bullet\}\}}}{\Gamma\{\Sigma\{\emptyset\},\Delta\{B^\bullet\}\}}}{\Gamma\{\Sigma\{\emptyset\},\Delta\{\emptyset\}\}}$$

where $\mathcal{D}_2'$ exists because our original derivation is left-free, and the w-step is removed by Proposition 6.8. Note that it can happen that $(\Sigma\{\emptyset\})^{\Downarrow} = (\Sigma\{A^\bullet\})^{\Downarrow}$ and/or $(\Delta\{\emptyset\})^{\Downarrow} = (\Delta\{B^\bullet\})^{\Downarrow}$, depending on where the output formula occurs in $\Gamma\{\Sigma\{\emptyset\},\Delta\{\emptyset\}\}$.

Above, we have only shown the cases for cut. The ones for $\Diamond\mathsf{cut}$ and $\Box\mathsf{cut}$ are similar, except the ones when r is one of $\mathsf{sb}^{[]}$, $\mathsf{s5b}^{[]}$, $\mathsf{s5}^{[]}$, or $\mathsf{sb5}^{[]}$. When such a rule is on the right above a $\Box\mathsf{cut}$, we decompose that $\Box\mathsf{cut}$ into a $\mathsf{s4}^\bullet$ and a cut (using Fact 6.15) and then apply Lemma 6.16 to permute the $\mathsf{s4}^\bullet$ over r, so that we can proceed as described above. When the cut is permuted over r, we can compose it again with the $\mathsf{s4}^\bullet$-instance, so that we can proceed by induction hypothesis. Observe that we make crucial use of the left-free property. Without it, the number of origins in $G(\mathcal{D})$ would not be stable. Furthermore, note that the cut-cut permutation does not affect cuts that are above our current origin. Thus, all origins above remain anchored. This is the reason for starting with the topmost one. ∎

Lemma 6.17. *Let $\langle\mathsf{X},\mathsf{Y}\rangle$ be a safe pair of axioms. If there is a proof*

$$\ast\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1}{\Gamma_1\{A^\circ\}} \quad \overset{\mathcal{D}_2}{\Gamma_2\{A^\bullet\}}}{\Gamma\{\emptyset\}} \tag{6.8}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are both in $\mathsf{NCK}' + \mathsf{X}_{\mathsf{s}}^{◐} + \mathsf{Y}_{\mathsf{s}}^{[]}$ and where $\ast\mathsf{cut}$ is anchored, then there is a proof $\mathcal{D}'$ of $\Gamma\{\emptyset\}$ in $\mathsf{NCK}' + \mathsf{X}_{\mathsf{s}}^{◐} + \mathsf{Y}_{\mathsf{s}}^{[]} + \mathsf{Cut}$ in which all cuts have a smaller rank.

Proof. We make a case analysis on the cut-formula A.

(1) If $A = B \wedge C$, we reduce the cut rank as follows:

$$\mathsf{cut}\,\dfrac{\wedge^\circ\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{B^\circ\}} \quad \overset{\mathcal{D}_1''}{\Gamma^{\Downarrow}\{C^\circ\}}}{\Gamma^{\Downarrow}\{B\wedge C^\circ\}} \qquad \wedge^\bullet\,\dfrac{\overset{\mathcal{D}_2'}{\Gamma\{B^\bullet, C^\bullet\}}}{\Gamma\{B\wedge C^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\wedge)}{\leadsto} \quad \mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{B^\circ\}} \qquad \mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1''}{\Gamma^{\Downarrow}\{C^\circ\}}}{\Gamma^{\Downarrow}\{B^\bullet, C^\circ\}} \quad \overset{\mathcal{D}_2'}{\Gamma\{B^\bullet, C^\bullet\}}}{\Gamma\{B^\bullet\}}}{\Gamma\{\emptyset\}}$$

where $\mathcal{D}_1'$ and $\mathcal{D}_1''$ exist by Proposition 6.9, and $\mathcal{D}_2'$ exists since our cut is anchored. Finally, we remove the w-step using Proposition 6.8.

(2) If $A = B \supset C$, we reduce the cut rank as follows:

$$\mathsf{cut}\,\dfrac{\supset^\circ\,\dfrac{\overset{\mathcal{D}'_1}{\Gamma^{\Downarrow}\{B^\bullet, C^\circ\}}}{\Gamma^{\Downarrow}\{B \supset C^\circ\}} \qquad \supset^\bullet\,\dfrac{\overset{\mathcal{D}'_2}{\Gamma^{\Downarrow}\{B^\circ\}} \quad \overset{\mathcal{D}''_2}{\Gamma\{C^\bullet\}}}{\Gamma\{B \supset C^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\supset)}{\leadsto} \quad \mathsf{cut}\,\dfrac{\overset{\mathcal{D}'_2}{\Gamma^{\Downarrow}\{B^\circ\}} \qquad \mathsf{cut}\,\dfrac{\overset{\mathcal{D}'_1}{\Gamma^{\Downarrow}\{B^\bullet, C^\circ\}} \quad \overset{\mathcal{D}''_2}{\Gamma\{C^\bullet\}}}{\Gamma\{B^\bullet\}}}{\Gamma\{\emptyset\}}$$

where $\mathcal{D}'_1$ exists by Proposition 6.9, and $\mathcal{D}'_2$ and $\mathcal{D}''_2$ exist since our cut is anchored.

(3) If $A = \Box B$, and the rule on the right above the cut is a $\Box^\bullet$, then we reduce the cut rank as follows:

$$\Box\mathsf{cut}\,\dfrac{\Box^\circ\,\dfrac{\overset{\mathcal{D}'_1}{\Gamma^{\Downarrow}\{[B^\circ], (\Theta\{[\Delta]\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{\Box B^\circ, (\Theta\{[\Delta]\})^{\Downarrow}\}} \qquad \Box^\bullet\,\dfrac{\overset{\mathcal{D}'_2}{\Gamma\{\Theta\{[B^\bullet, \Delta]\}\}}}{\Gamma\{\Theta\{\Box B^\bullet, [\Delta]\}\}}}{\Gamma\{\Theta\{[\Delta]\}\}} \quad \overset{(\Box)}{\leadsto}$$

$$\mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{(n{+}1){*}\mathsf{w}\,\dfrac{\mathsf{w}\,\dfrac{n{*}\mathsf{4}^{[]}\,\dfrac{\overset{\mathcal{D}'_1}{\Gamma^{\Downarrow}\{[B^\circ], (\Theta\{[\Delta]\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{[\![[B^\circ]]\!]^n, (\Theta\{[\Delta]\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{[\![[B^\circ, \Delta^{\Downarrow}]]\!]^n, (\Theta\{[\Delta]\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{\Theta^{\Downarrow}\{[B^\circ, \Delta^{\Downarrow}]\}, (\Theta\{[\Delta]\})^{\Downarrow}\}} \qquad \mathsf{w}\,\dfrac{\overset{\mathcal{D}'_2}{\Gamma\{\Theta\{[B^\bullet, \Delta]\}\}}}{\Gamma\{\Theta\{[B^\bullet, \Delta]\}, (\Theta\{[\Delta]\})^{\Downarrow}\}}}{\Gamma\{\Theta\{[\Delta]\}, (\Theta\{[\Delta]\})^{\Downarrow}\}}}{\Gamma\{\Theta\{[\Delta]\}\}}$$

Where $\mathcal{D}'_1$ and $\mathcal{D}'_2$ exist for the same reason as above, and we finally apply Proposition 6.8 to remove the w- and $\mathsf{4}^{[]}$-steps, where n is the depth of $\Theta^{\Downarrow}\{\ \}$.

(4) If $A = \Box B$, and the rule on the right above the cut is a $\mathsf{t}^\bullet$, then we reduce the cut rank as follows:

$$\Box\mathsf{cut}\,\dfrac{\Box^\circ\,\dfrac{\overset{\mathcal{D}'_1}{\Gamma^{\Downarrow}\{[B^\circ], (\Theta\{\emptyset\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{\Box B^\circ, (\Theta\{\emptyset\})^{\Downarrow}\}} \qquad \mathsf{t}^\bullet\,\dfrac{\overset{\mathcal{D}'_2}{\Gamma\{\Theta\{B^\bullet\}\}}}{\Gamma\{\Theta\{\Box B^\bullet\}\}}}{\Gamma\{\Theta\{\emptyset\}\}} \quad \overset{(\Box\mathsf{t})}{\leadsto} \quad \mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{(n{+}1){*}\mathsf{w}\,\dfrac{(n{-}1){*}\mathsf{4}^{[]}/\mathsf{t}^{[]}\,\dfrac{\overset{\mathcal{D}'_1}{\Gamma^{\Downarrow}\{[B^\circ], (\Theta\{\emptyset\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{[\![B^\circ]\!]^n, (\Theta\{\emptyset\})^{\Downarrow}\}}}{\Gamma^{\Downarrow}\{\Theta^{\Downarrow}\{B^\circ\}, (\Theta\{\emptyset\})^{\Downarrow}\}} \qquad \mathsf{w}\,\dfrac{\overset{\mathcal{D}'_2}{\Gamma\{\Theta\{B^\bullet\}\}}}{\Gamma\{\Theta\{B^\bullet\}, (\Theta\{\emptyset\})^{\Downarrow}\}}}{\Gamma\{\Theta\{\emptyset\}, (\Theta\{\emptyset\})^{\Downarrow}\}}}{\Gamma\{\Theta\{\emptyset\}\}}$$

where n is the depth of $\Theta^{\Downarrow}\{\ \}$, and at the top we either have one $\mathsf{t}^{[]}$ step (if $n = 0$) or $n - 1$ steps of $\mathsf{4}^{[]}$ (if $n \geq 1$), that can be removed via Proposition 6.8. Note that $\mathsf{4} \in \mathsf{X}$ if $n \geq 1$.

(5) If $A = a$, then the cut is removed as follows:

$$\mathsf{cut}_1\,\dfrac{\overset{\mathcal{D}_1}{\Gamma\{a^\circ\}} \qquad \mathsf{id}\,\dfrac{}{\Gamma\{a^\bullet, a^\circ\}}}{\Gamma\{a^\circ\}} \quad \overset{(\mathsf{id})}{\leadsto} \quad \overset{\mathcal{D}_1}{\Gamma\{a^\circ\}}$$

Note that that here $\Gamma^{\Downarrow}\{a^\circ\} = \Gamma\{a^\circ\}$.

(6) If $A = \bot$, the situation is similar:

$$\mathsf{cut}_1 \dfrac{\dfrac{\mathcal{D}_1}{\Gamma\{\Pi^\circ\}} \qquad \bot^\bullet \dfrac{}{\Gamma\{\bot^\bullet, \Pi^\circ\}}}{\Gamma\{\Pi^\circ\}} \quad \overset{(\bot)}{\leadsto} \quad \dfrac{\mathcal{D}_1}{\Gamma\{\Pi^\circ\}}$$

(7) If $A = B \vee C$, we proceed by induction on the height of $\mathcal{D}_1$ and make a case analysis on its bottommost rule instance r.

(a) If r is a $\vee^\circ$, then it has to decompose the cut formula, and we can reduce the cut rank as follows:

$$\mathsf{cut} \dfrac{\vee^\circ \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{B^\circ\}}}{\Gamma^{\Downarrow}\{B \vee C^\circ\}} \qquad \vee^\bullet \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma\{B^\bullet\}} \quad \dfrac{\mathcal{D}_2''}{\Gamma\{C^\bullet\}}}{\Gamma\{B \vee C^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\vee)}{\leadsto} \quad \mathsf{cut} \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{B^\circ\}} \qquad \dfrac{\mathcal{D}_2'}{\Gamma\{B^\bullet\}}}{\Gamma\{\emptyset\}}$$

There is a similar case where $\vee^\circ$ chooses C°.

(b) If r is a $\supset^\bullet$, then we have

$$\mathsf{cut} \dfrac{\supset^\bullet \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{D^\circ\}\}} \quad \dfrac{\mathcal{D}_1''}{\Gamma^{\Downarrow}\{\Theta^\bullet\{E^\bullet\}, \Delta^{\Downarrow}\{A^\circ\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{D \supset E^\bullet\}, \Delta^{\Downarrow}\{A^\circ\}\}} \qquad \dfrac{\mathcal{D}_2'}{\Gamma\{\Theta^\bullet\{D \supset E^\bullet\}, \Delta\{A^\bullet\}\}}}{\Gamma\{\Theta^\bullet\{D \supset E^\bullet\}, \Delta\{\emptyset\}\}} \quad \overset{(\supset^\bullet)}{\leadsto}$$

$$\supset^\bullet \dfrac{\mathsf{w} \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{D^\circ\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{D^\circ\}, (\Delta\{\emptyset\})^{\Downarrow}\}} \qquad \mathsf{cut} \dfrac{\dfrac{\mathcal{D}_1''}{\Gamma^{\Downarrow}\{\Theta^\bullet\{E^\bullet\}, \Delta^{\Downarrow}\{A^\circ\}\}} \quad \mathsf{Inv} \dfrac{\dfrac{\mathcal{D}_2'}{\Gamma\{\Theta^\bullet\{D \supset E^\bullet\}, \Delta\{A^\bullet\}\}}}{\Gamma\{\Theta^\bullet\{E^\bullet\}, \Delta\{A^\bullet\}\}}}{\Gamma\{\Theta^\bullet\{E^\bullet\}, \Delta\{\emptyset\}\}}}{\Gamma\{\Theta^\bullet\{D \supset E^\bullet\}, \Delta\{\emptyset\}\}}$$

where the Inv-step is removed by Proposition 6.9, and we can proceed by induction hypothesis.

(c) All other cases are handled symmetric to their corresponding cases in Lemma 6.12. This, in particular, concerns the case where r is $\vee^\bullet$. Since our cut is anchored, the output-branch of the sequent is next to the $A^\bullet$ in the right premise of the cut. Therefore, the $\vee^\bullet$ above the left premise of the cut can now be permuted under the cut:

$$\mathsf{cut} \dfrac{\vee^\bullet \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1\{A^\circ\}} \quad \dfrac{\mathcal{D}_1''}{\Gamma_2\{A^\circ\}}}{\Gamma\{A^\circ\}} \qquad \dfrac{\mathcal{D}_2}{\Gamma\{A^\bullet, \Pi^\circ\}}}{\Gamma\{\Pi^\circ\}} \quad \overset{(\vee^\bullet)}{\leadsto}$$

$$\vee^\bullet \dfrac{\mathsf{cut} \dfrac{\dfrac{\mathcal{D}_1'}{\Gamma_1\{A^\circ\}} \quad \mathsf{Inv} \dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{A^\bullet, \Pi^\circ\}}}{\Gamma_1\{A^\bullet, \Pi^\circ\}}}{\Gamma_1\{\Pi^\circ\}} \qquad \mathsf{cut} \dfrac{\dfrac{\mathcal{D}_1''}{\Gamma_2\{A^\circ\}} \quad \mathsf{Inv} \dfrac{\dfrac{\mathcal{D}_2}{\Gamma\{A^\bullet, \Pi^\circ\}}}{\Gamma_2\{A^\bullet, \Pi^\circ\}}}{\Gamma_2\{\Pi^\circ\}}}{\Gamma\{\Pi^\circ\}}$$

The Inv-steps are removed by Proposition 6.9. The other invertible rules are handled similarly:

$$\mathsf{cut}\,\dfrac{\mathsf{r}\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma_1^{\Downarrow}\{A^\circ\}}}{\Gamma^{\Downarrow}\{A^\circ\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}} \quad \overset{(\mathsf{r})}{\leadsto} \quad \mathsf{r}\,\dfrac{\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma_1^{\Downarrow}\{A^\circ\}} \qquad \mathsf{Inv}\,\dfrac{\overset{\mathcal{D}_2}{\Gamma\{A^\bullet\}}}{\Gamma_1\{A^\bullet\}}}{\Gamma_1\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

(8) If $A = \Diamond B$ we proceed as in the previous case. The only difference occurs when the bottommost rule r in $\mathcal{D}_1$ works on A°. There are three subcases:

(a) If r is $\mathsf{s4}^\circ$ we have

$$\Diamond\mathsf{cut}\,\dfrac{\mathsf{s4}^\circ\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{\Delta^\bullet\{\Diamond B^\circ\}\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{\Diamond B^\circ, \Delta^\bullet\{\emptyset\}\}\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond B^\bullet, \Theta^\bullet\{\Delta^\bullet\{\emptyset\}\}\}}}{\Gamma\{\Theta^\bullet\{\Delta^\bullet\{\emptyset\}\}\}} \quad \overset{(\Diamond 4)}{\leadsto} \quad \Diamond\mathsf{cut}\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{\Delta^\bullet\{\Diamond B^\circ\}\}\}} \qquad \overset{\mathcal{D}_2}{\Gamma\{\Diamond B^\bullet, \Theta^\bullet\{\Delta^\bullet\{\emptyset\}\}\}}}{\Gamma\{\Theta^\bullet\{[\Delta^\bullet\{\emptyset\}]\}\}}$$

and can proceed by the induction hypothesis.

(b) If r is $\Diamond^\circ$ we can reduce the cut rank as follows:

$$\Diamond\mathsf{cut}\,\dfrac{\Diamond^\circ\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{[B^\circ, \Delta^\bullet]\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{\Diamond B^\circ, [\Delta^\bullet]\}\}} \qquad \Diamond^\bullet\,\dfrac{\overset{\mathcal{D}_2'}{\Gamma\{[B^\bullet], \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{\Diamond B^\bullet, \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{\Theta^\bullet\{[\Delta^\bullet]\}\}} \quad \overset{(\Diamond)}{\leadsto}$$

$$\mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{[B^\circ, \Delta^\bullet]\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{[B^\circ, \Delta^\bullet]\}, \Theta^\bullet\{[\Delta^\bullet]\}\}} \qquad {\scriptstyle (n+1)*}\mathsf{w}\,\dfrac{\mathsf{w}\,\dfrac{{\scriptstyle n*}\mathsf{4}^{[]}\,\dfrac{\overset{\mathcal{D}_2'}{\Gamma\{[B^\bullet], \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{[\![[B^\bullet]]\!]^n, \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{[\![[B^\bullet, \Delta^\bullet]]\!]^n, \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{\Theta^\bullet\{[B^\bullet, \Delta^\bullet]\}, \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{\Theta^\bullet\{[\Delta^\bullet]\}, \Theta^\bullet\{[\Delta^\bullet]\}\}}}{\Gamma\{\Theta^\bullet\{[\Delta^\bullet]\}\}}$$

where n is the depth of the context $\Theta^\bullet\{\ \}$, and $\mathcal{D}_2'$ exists because the instance of $\Diamond\mathsf{cut}$ is anchored. We use Proposition 6.8 to remove the w- and $\mathsf{4}^{[]}$-steps. We proceed similarly for $\mathsf{s4}^\circ_\Diamond$.

(c) If r is t°, the situation is similar, and we can reduce the cut rank as follows:

$$
\Diamond\mathsf{cut}\,\dfrac{\mathsf{t}^\circ\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{B^\circ\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{\Diamond B^\circ\}\}} \qquad \Diamond^\bullet\,\dfrac{\overset{\mathcal{D}_2'}{\Gamma\{[B^\bullet],\Theta^\bullet\{\emptyset\}\}}}{\Gamma\{\Diamond B^\bullet,\Theta^\bullet\{\emptyset\}\}}}{\Gamma\{\Theta^\bullet\{\emptyset\}\}} \quad \overset{(\Diamond\mathsf{t})}{\leadsto}
$$

$$
\mathsf{c}\,\dfrac{\mathsf{cut}\,\dfrac{\mathsf{w}\,\dfrac{\overset{\mathcal{D}_1'}{\Gamma^{\Downarrow}\{\Theta^\bullet\{B^\circ\}\}}}{\Gamma^{\Downarrow}\{\Theta^\bullet\{B^\circ\},\Theta^\bullet\{\emptyset\}\}} \qquad (n{+}1){*}\mathsf{w}\,\dfrac{(n{-}1){*}\mathsf{4}^{[]}/\mathsf{t}^{[]}\,\dfrac{\overset{\mathcal{D}_2'}{\Gamma\{[B^\bullet],\Theta^\bullet\{\emptyset\}\}}}{\Gamma\{[\![B^\bullet]\!]^n,\Theta^\bullet\{\emptyset\}\}}}{\Gamma\{\Theta^\bullet\{B^\bullet\},\Theta^\bullet\{\emptyset\}\}}}{\Gamma\{\Theta^\bullet\{\emptyset\},\Theta^\bullet\{\emptyset\}\}}}{\Gamma\{\Theta^\bullet\{\emptyset\}\}}
$$

Again, n is the depth of the context $\Theta^\bullet\{\ \}$. If $n = 0$, there are no brackets, and we use $\mathsf{t}^{[]}$. If $n \geq 1$, there is at least one bracket nesting (and therefore $\mathsf{4} \in \mathsf{X}$), and we use $n - 1$ instances of $\mathsf{4}^{[]}$, which then are removed by applying Proposition 6.8. ☐

Theorem 6.18. *Let $\langle \mathsf{X}, \mathsf{Y}\rangle$ be a safe pair of axioms, and let $\mathcal{D}$ be a proof in $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]} + \mathsf{Cut}$. Then there is a proof $\mathcal{D}'$ of the same conclusion in $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$.*

Proof. A proof in $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]} + \mathsf{Cut}$ is trivially also a proof in $\mathsf{NCK}' + \mathsf{X}^{◐}_{\mathsf{s}} + \mathsf{Y}^{[]}_{\mathsf{s}} + \mathsf{Cut}$. We proceed by induction on the cut-value $\boldsymbol{v}(\mathcal{D})$. We pick a topmost cut in $\mathcal{D}$. If this $*\mathsf{cut}$-instance is anchored, then we can by Lemma 6.17 replace this cut by cuts of smaller rank, and thus reduce the overall cut-value of the derivation. If our $*\mathsf{cut}$-instance is not anchored, we observe that the subderivation rooted at that $*\mathsf{cut}$-instance is left-free (because we chose a topmost cut), and therefore we can apply Lemma 6.12 to replace that subderivation with one in which all cuts are anchored and have the same rank. Thus, the overall cut-value of the derivation has reduced as well, and we can proceed by the induction hypothesis. Finally, we apply Proposition 6.3 to get a proof in $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$. ☐

From here it is simple to see why our main theorem, 5.2, holds.

Proof of Thm. 5.2. Since Cut always contains cut we have that $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]} + \mathsf{Cut}$ is complete with respect to $\mathsf{HCK} + \mathsf{X} + \mathsf{Y}$ by Thm. 5.1, and by the discussion on safety at the beginning of Sect. 6, namely Prop. 6.1. The completeness of $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$ then follows by Thm. 6.18 above. ☐

7. Conclusions

We presented a unified framework for the constructive modal cube. To the best of our knowledge, this is the first attempt to provide such a framework.

Even though we did not manage to prove cut-elimination for all logics, we conjecture that our systems admit cut. More precisely:

Conjecture 7.1. *Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{4}\}$ and $\mathsf{Y} \subseteq \{\mathsf{d}, \mathsf{b}, \mathsf{5}\}$, such that if $\mathsf{t} \in \mathsf{X}$ and $\mathsf{5} \in \mathsf{Y}$ then $\mathsf{b} \in \mathsf{Y}$. Then every theorem of $\mathsf{CK} + \mathsf{X} + \mathsf{Y}$ is provable in $\mathsf{NCK}' + \mathsf{X}^{◐} + \mathsf{Y}^{[]}$.*

This would give us a cut-free system for every logic in the cube. The reason why we think Conjecture 7.1 is true is the observation that the only place where $\mathsf{4}$ steps appear in the presence of $\mathsf{b}^{[]}$ or $\mathsf{5}^{[]}$ is the permutation of $\mathsf{sb}^{[]}$ steps or $\mathsf{s5}^{[]}$ steps under a cut, as in (6.5). Then instances of $\mathsf{4}^{[]}$ are introduced, and in the admissibility proof for $\mathsf{4}^{[]}$, instances of $\mathsf{4}^{\bullet}$ or $\mathsf{4}^{\circ}$ are introduced only when the $\mathsf{4}^{[]}$ step is permuted over instances of $\Box^{\bullet}$ or $\Diamond^{\circ}$. But, looking back at (6.5) and (6.7), one can see that it seems possible to permute these instances of $\Box^{\bullet}$ and $\Diamond^{\circ}$ under the whole derivation block, including the cut. However, we have not yet managed to incorporate this observation into the formal cut-elimination argument, and leave this issue for further research.

Another path of further research is to give modal logics a similar uniform treatment as the substructural logics in [CGT08, CST09]. For this, it is necessary to first look at concrete examples of structural rules corresponding to axioms, as we have shown in Figure 4. Since these rules almost coincide in the classical, the intuitionistic, and the constructive setting, we hope to eventually discover a general pattern, yielding uniform cut-elimination arguments for a variety of modal logics.

Finally, we have observed an apparent dichotomy between the b axiom and the 'constructiveness' of constructive modal logic, since the former implies k_3 and k_5, for which we do not know of any approach providing some sort of Curry-Howard correspondence. We therefore believe it would be pertinent to develop further outlooks on these logics, for example a Kripke semantics, in order to understand their role better; this could also form the basis of future work.

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