A Representation of Proofs with Cut as Higher Order Recursion Schemes

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The property of being a valid first-order formula is intimately tied to the consideration of the ground, i.e., variable-free, instances of that formula. This connection is apparent in most, if not all, proofs of the completeness theorem which, in one way or another, rely on the construction of a term model. It is plainly visible in Herbrand's theorem which states that a formula is valid if, and only if, there is a finite expansion (of existential quantifiers to disjunctions and universal quantifiers to conjunctions of instances). This feature of classical first-order logic is in contrast to both classical second-order logic, whose standard semantics goes beyond the ground instances of a countable language, and intuitionistic first-order logic, which exhibits a more complicated interaction between quantifiers and propositional connectives.

Proof-theoretically, the use of instances of a formula naturally leads to analytic, cut-free, proofs. Gentzen's mid-sequent theorem makes the close connection between Herbrand expansions and cut-free proofs clearly visible. Taking this perspective on the cut-elimination theorem, and thereby keeping the well-known complexity bounds in mind, shows that, in essence, cut-elimination consists of the computation of a Herbrand expansion. One may ask, however, whether given a proof with cut it is possible to compute a Herbrand expansion in a more direct way, circumventing the cumbersome process of cut-elimination. Indeed, there is a number of formalisms that do just that, the historically first being Hilbert's ε-calculus [16]. In [7], Gerhardy and Kohlenbach adapt Shoenfield's variant of Gödel's Dialectica interpretation [20] to a system of pure predicate logic. Recent work, related to proof nets, is that of Heijltjes [8] and McKinley [18], and a similar approach, in the formalism of expansion trees can be found in [15]. A different method with similar aims is cut-elimination by resolution [4].

The present work is motivated by the follow-up question: what is a minimal amount of information required for computing a Herbrand expansion from a proof with cuts? An approach which has been partially successful in answering the question is the representation of proofs as tree grammars, introduced in [9] for proofs with Π_1-cuts and extended to Π_2-cuts in [1, 2]. This emphasis on minimality plays a crucial role for several applications, such as cut-introduction [12, 11, 17], inductive theorem proving [5] and the confluence behaviour of cut-elimination [13, 14, 2]. Algorithms for cut-introduction and inductive theorem proving are currently being implemented in the GAPT-system [6], see e.g. [10].

Continuing this research effort, in this talk we demonstrate how Herbrand expansions can be represented as higher order recursion schemes derived directly from first-order proofs with cut. Higher order recursion schemes (see e.g. [19]) are a generalisation of regular tree grammars (which correspond to order-0 recursion schemes) to finite types. The representation we outline involves interpreting inference rules of proofs as non-terminals whose production rules follow the local instantiation structure of quantifiers. The type of a non-terminal is determined entirely by the quantifier complexity of the formulæ occurring in the corresponding inference, with an inference deriving $\Sigma_n \cup \Pi_n$ formulæ being represented by a non-terminal of order n. Cut corresponds to composition of non-terminals, and instances of contraction give rise to nondeterministic production rules. The language of the recursion scheme induces a Herbrand expansion for the end-sequent of the proof. At the level of Π_2-cuts, the schemes closely resemble the grammars introduced in [3]; the generic case of the representation that permits capturing cuts of arbitrary complexity turns out to be at the level of Π_3 where both sides of a cut feature $\exists\forall$ quantifier alternations.

As far as we are aware, the present work marks the first method of Herbrand extraction that operates directly on sequent calculus proofs. The main result can be summarised as follows.

Theorem. *Let F be a quantifier-free formula and π an LK-proof of $\exists\vec{v}F$ in which cut-formulæ are prenex Π_n or Σ_n. There exists an acyclic order n recursion scheme $\mathscr{H}$ with language $L(\mathscr{H})$ such that: i) $\bigvee_{\vec{t}\in L(\mathscr{H})} F(\vec{t})$ is valid; ii) $|L(\mathscr{H})| \leq 2^{4|\pi|^4}_{n+2}$ where $|\pi|$ is the number of inference rules in π; iii) $L(\mathscr{H})$ contains the Herbrand set extracted from any cut-free proof that can be obtained from π via a sequence of Gentzen-style cut reductions that always reduces to the weak (quantifier) side of a cut before the strong side.*

There are two points of note regarding the theorem. The first is the restriction to cuts on prenex formulæ. Our presentation assumes a Tait-style (one-sided) sequent calculus for first-order logic that necessitates an asymmetric interpretation of formulæ to types which does not generalise to non-prenex cuts (existentially quantified formulæ force a type order one higher than universally quantified formulæ of the same depth). It is our expectation, however, that a reformulation of the recursion scheme suited to the Gentzen-style sequent calculus will permit a uniform association of formulæ to types that allows for a generalisation to cuts of any form while maintaining the same bounds. The second point concerns the upper bound on the size of the Herbrand expansion which is, roughly, a single exponential larger than the upper bounds computed via other methods. Nevertheless, the bound may still prove optimal for the theorem as stated given that the language $L(\mathscr{H})$ includes the Herbrand expansion of (exponentially) many different cut-free proofs, whereas other approaches focus on the computation of a single 'minimal' Herbrand expansion.

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