Stoic Analysis

Susanne Bobzien[1] and Roy Dyckhoff[2]

[1] Oxford University, UK
susanne.bobzien@philosophy.ox.ac.uk

[2] St Andrews University, UK
rd@st-andrews.ac.uk

Abstract

Stoic logic is considered as a candidate for an early example of structural proof theory; in particular, we consider the extent to which one may show a "normal form" theorem for Stoic proofs.

1 Introduction

In his first paper, Gentzen showed [3] a normal form theorem for a variant of a calculus [5] of Hertz; proofs could be transformed so that (i) the only instance (if any) of *Thinning*[1] was the final step and (ii) the first premiss of any instance of *Cut* was an initial sequent (i.e. either a *tautology* $A \Rightarrow A$ or one of the assumption sequents on which the proof depends). This is a calculus in which cuts are not eliminable: logical symbols are given meaning by the presence of assumption sequents such as $A \wedge B \Rightarrow A$ rather than by rules such as Gentzen's $\&{-}IA$, "Introduction of & in the Antecedent" [4].

Much further back in time, the Stoics worked [1, 2] with sequents (for details see below) and depended for completeness (in some sense not yet made clear) on various *Cut rules*. As in [3, 5], logical symbols are given meaning not by rules but by axioms. Our interest here is the extent to which there is a normal form theorem like that of [3]; the question of how much the Stoics understood this is secondary (and not easily answered given the fragmentary nature of the sources). We give an example refuting such a theorem and speculate on how minor changes to our hypothetical version of the Stoic system might yet allow its proof.

2 Definitions

We refer to [1] and [2]; however, modern standard proof-theoretic notation and terminology are used (where possible). *Formulae* $A, B, \ldots$ are either atomic or non-atomic; the *atomic* ones (i.e. *atoms*) are just the (positive) literals $p, q, r, \ldots, p', q', r', \ldots, \ldots$. The atoms $p, q, r \ldots$ appearing in examples are distinct. *Non-atomic* formulae are built up from atoms using *negation* ($\neg A$), *conjunction* ($A \wedge B$), *exclusive disjunction* ($A \oplus B$) and *conditional* ($A \rightarrow B$). $\wedge$ and $\oplus$ are left-associative; $\rightarrow$ is right-associative. Conventions such as left and right associativity are ours, to simplify our exposition, rather than Stoic. The Stoics would not have distinguished between $A \wedge B$ and $B \wedge A$; and similarly for $A \oplus B$.

The *contradictory* A^* of a formula A is defined using $(\neg B)^* = B$ and otherwise $A^* = \neg A$.

Sets Γ of formulae are as usual; a comma indicates the construction of the union of two sets. The *empty set* is allowed; it is written $\emptyset$ or omitted.

A *sequent* is of the form $\Gamma \vdash A$, where Γ is a *Stoic context*, i.e. $|\Gamma| > 1$. We will omit the word "Stoic" henceforth. The sequent $\Gamma \vdash A$ has Γ as its *antecedent* and A as its *succedent*:

[1] aka *Weakening*

members of Γ are its *antecedent formulae*. In each of the rules $T1, \ldots$ below, the sequents above the line are its *premisses*; the sequent below the line is its *conclusion*. *Derivations* are finite trees constructed in the usual way using the axioms and rules. A sequent is *derivable* iff it is the end-sequent of a derivation. It is implicit in the statement of the rules $T1, \ldots$ that the conclusion is not an axiom. We make no claim that such trees are historical.

The (schematic) *axioms* are as follows:

$$\frac{}{A \rightarrow B, A \vdash B}\ A1 \qquad \frac{}{A \rightarrow B, B^* \vdash A^*}\ A2 \qquad \frac{}{\neg(A \wedge B), A \vdash B^*}\ A3$$

$$\frac{}{A \oplus B, A \vdash B^*}\ A4 \qquad \frac{}{A \oplus B, B^* \vdash A}\ A5$$

and the *rules* are as follows:

$$\frac{\Gamma, A \vdash B}{\Gamma, B^* \vdash A^*}\ T1 \qquad \frac{A, B \vdash C \quad C, \Gamma \vdash D}{A, B, \Gamma \vdash D}\ T3$$

$$\frac{A, B \vdash C \quad C, A \vdash D}{A, B \vdash D}\ T2 \qquad \frac{A, B \vdash C \quad C, A, B \vdash D}{A, B \vdash D}\ T2'$$

$$\frac{A, B \vdash C \quad C, A, \Gamma \vdash D}{A, B, \Gamma \vdash D}\ T4 \quad (\Gamma \neq \emptyset) \qquad \frac{A, B \vdash C \quad C, A, B, \Gamma \vdash D}{A, B, \Gamma \vdash D}\ T4' \quad (\Gamma \neq \emptyset)$$

The axioms are the five kinds of "indemonstrable" from [2, pp 104–105]. The rules (known to the Stoics as "themata") are from [2, pp 114–117], based either on Stoic sources or reconstructions thereof.

3 Notes on definitions

Note that $p \vdash p$ isn't an axiom: it isn't even a sequent. Also, even though $p, q \vdash p$ is a sequent, it isn't an axiom. As we will see, it is underivable. Intuitions from classical, intuitionistic or minimal logic must be discarded: this logic is a substructural logic. However, if a sequent is derivable, then it is derivable in classical logic, so we can easily reject some sequents as underivable.

The combination of $A3$ with $T1$ provides a derived rule similar to Gentzen's $R\wedge$ (we use notation from [6]); but there is no rule comparable to $L\wedge$. The Stoics would have distinguished between $A, B, \Gamma \vdash C$ and $A \wedge B, \Gamma \vdash C$. Likewise, there is no rule comparable to Gentzen's rule $R\rightarrow$ introducing $\rightarrow$ on the right; and the Stoics would have distinguished between $\Gamma \vdash A \rightarrow B$ and $\Gamma, A \vdash B$ (assuming that Γ is large enough to allow acceptance as a Stoic sequent).

The restriction that $\Gamma \neq \emptyset$ in $T4$ (and $T4'$) reflects "one (or more)" as written[2] in [2, p 117].

There is a crucial distinction from Gentzen's sequent calculus **LJ** [4]: the logical symbols are (as in Frege-Hilbert systems) axiomatised, whereas for Gentzen [4] their meanings were incorporated into the inference rules, such as $L\rightarrow$ and $R\wedge$. This feature of Gentzen's system allowed the proof of the "Hauptsatz", i.e. that every cut could be eliminated. The cuts in the Stoic system are in the rules $T2, T2', T3, T4, T4'$ and are not eliminable.

We can suppose that the last five rules ($T2, T2', T3, T4, T4'$) are replaced by just **one** rule:

$$\frac{A, B \vdash C \quad C, [A, B], \Gamma \vdash D}{A, B, \Gamma \vdash D}\ S$$

[2] $T4$ is expressed as "When from two assertibles a third follows, and from the third and one (or both) of the two and one (or more) external assertible(s) another follows, then this other follows from the first two and the external(s)".

in which $[A, B]$ indicates that neither, one or both of A, B is required—each is optional. There is a Stoic precedent for this form of the rule.

We may also use a restricted form of S, the restriction being that the first premiss **must** be an axiom:

$$\frac{\overline{A, B \vdash C} \qquad C, [A, B], \Gamma \vdash D}{A, B, \Gamma \vdash D} \; RS$$

This being the case, it is clear (from the forms of the axioms) that A and B are automatically distinct.

4 The search space and proof search

A successive pair of $T1$ instances can always be reduced to 0 or 1. The calculus consisting of axioms and rules $T1, S$ is not analytic, and without some form of loop-checking depth-first root-first search may fail to terminate. The rule of *Weakening* is not admissible: this is a substructural logic. None of the rules other than $T1$ is invertible. Decidability is an open problem: the following result (partially expressing relevance) is useful:

Theorem If an atom occurs positively (negatively) in a derivable sequent, then it also has a negative (resp., positive) occurrence therein. □

It may be conjectured that there is a normal form for proofs, i.e. that the only instances of S required are those of RS, i.e. with the first premiss an axiom; this significantly restricts proof search and ensuring analyticity in some sense. The following sequent, however, has a derivation using S but not one using RS:

$$p, q, r, s \vdash (p \wedge q) \wedge (r \wedge s)$$

Work is therefore in progress trying to clarify what changes (not necessarily realised by the Stoics) can be made to allow a normal form theorem offering adequate analyticity and thus decidability. One option (not yet either accepted or ruled out) is to consider generalising the notion of axiom to those sequents that are derivable without use of the rule S (or any T-rule except $T1$). Sequents of the form $A, B \vdash A \wedge B$ are examples.

References

[1] S. Bobzien. Stoic syllogistic. In *Oxford Studies in Ancient Philosophy*, volume 14, pages 133–192. OUP, 1996.

[2] S. Bobzien. Stoic logic. In Brad Inwood, editor, *The Cambridge Companion to Stoic Philosophy*, pages 85–123. CUP, 2003.

[3] G. Gentzen. Über die Existenz unabhängiger Axiomensysteme zu unendlichen Satzsystemen. *Mathematische Annalen*, 107:329–350, 1932.

[4] G. Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1935.

[5] H. Hertz. Über Axiomensysteme für beliebige Satzsysteme. *Mathematische Annalen*, 101, 1929.

[6] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory (2nd edition).* CUP, 2000.