Intuitionistic modal logic in indexed nested sequents

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Modal logics were originally defined in terms of axioms in a Hilbert system, and later in terms of their semantics in relational structures. Structural proof theory for modal logics, however, was considered a difficult topic as traditional (Gentzen) sequents did not provide fully satisfactory (i.e. analytic and modular) proof systems even for some common modal logics.

Nonetheless, the proof theory of modal logics has received more attention in the last two decades, and some extensions of traditional sequents were successfully proposed to handle modalities. For example, *nested sequents* [4, 9, 1] are an extension of ordinary sequents to a structure of tree that has shown fruitful in providing proof systems for some modal logics.

However, the tree structure restricts the expressivity of nested sequents; in particular, it seems that they cannot give deductive systems for logics obeying the *Scott-Lemmon axioms*, which correspond semantically to a "confluence" condition on the relational structure [5].

Fitting recently introduced *indexed nested sequents* [3], an extension of nested sequents which goes beyond the tree structure to provide a proof system for classical modal logic K extended with an arbitrary set of Scott-Lemmon axioms. In this abstract we present our study of Fitting's system adapted to the intuitionistic case.

This is a part of the paper [7] that will be presented at *Tableaux'17*.

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We consider formulas from the following grammar, which extends the language of intuitionistic propositional logic with the two modalities $\Box$ and $\Diamond$:

$$A ::= a \mid \bot \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A \tag{1}$$

where a is taken from a countable set of propositional atoms.

Intuitionistic modal logic IK[1] is obtained from intuitionistic propositional logic by adding the axioms:

$$\begin{array}{ll} \mathsf{k}_1\colon\ \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3\colon\ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2\colon\ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4\colon\ (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & \mathsf{k}_5\colon\ \Diamond\bot \supset \bot \end{array} \tag{2}$$

and the *necessitation rule* nec that allows to derive the formula $\Box A$ from any theorem A.

Stronger modal logics can be obtained by adding to IK other axioms. In this work we are interested specifically in the family of *Scott-Lemmon axioms* of the form

$$\mathsf{g}_{\mathsf{klmn}}\colon (\Diamond^k \Box^l A \supset \Box^m \Diamond^n A) \wedge (\Diamond^m \Box^n A \supset \Box^k \Diamond^l A) \tag{3}$$

for a tuple (k, l, m, n) of natural numbers, where $\Box^m$ stands for m boxes and $\Diamond^n$ for n diamonds.

$$\bot^\bullet \, \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id} \, \frac{}{\Gamma\{a^\bullet, a^\circ\}} \qquad \top^\circ \, \frac{}{\Gamma\{\top^\circ\}}$$

$$\wedge^\bullet \, \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ \, \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \qquad \vee^\bullet \, \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ_1 \, \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \vee^\circ_2 \, \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\supset^\bullet \, \frac{\Gamma^*\{A \supset B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ \, \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet \, \frac{\Gamma\{\Box A^\bullet, [^{w} A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [^{w} \Delta]\}} \qquad \Box^\circ \, \frac{\Gamma\{[^{v} A^\circ]\}}{\Gamma\{\Box A^\circ\}} \qquad \Diamond^\bullet \, \frac{\Gamma\{[^{v} A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond^\circ_i \, \frac{\Gamma\{[^{w} A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [^{w} \Delta]\}}$$

$$\mathsf{tp} \, \frac{\Gamma\,^{w}\{\emptyset\}\,^{w}\{A\}}{\Gamma\,^{w}\{A\}\,^{w}\{\emptyset\}} \qquad \mathsf{bc}_1 \, \frac{\Gamma\,^{w}\{[^{u}\Delta]\}\,^{w}\{[^{u}\emptyset]\}}{\Gamma\,^{w}\{[^{u}\Delta]\}\,^{w}\{\emptyset\}} \qquad \mathsf{bc}_2 \, \frac{\Gamma\,^{w}\{[^{u}\Delta\,^{w}\{[^{u}\emptyset]\}]\}}{\Gamma\,^{w}\{[^{u}\Delta\,^{w}\{\emptyset\}]\}}$$

Figure 1: System iIKN

$$\mathsf{g_{klmn}} \, \frac{\Gamma\,^{u_0}\{[^{u_1}\Delta_1, \ldots [^{u_k}\Delta_k, [^{v_1} \ldots [^{v_l}] \ldots]] \ldots], [^{w_1}\Sigma_1, \ldots [^{w_m}\Sigma_m, [^{x_1} \ldots [^{x_n}] \ldots]] \ldots]\}}{\Gamma\,^{u_0}\{[^{u_1}\Delta_1, \ldots [^{u_k}\Delta_k] \ldots], [^{w_1}\Sigma_1, \ldots [^{w_m}\Sigma_m] \ldots]\}}$$

Figure 2: Inference rule $\mathsf{g_{klmn}}$ (where $k, l, m, n \neq 0$, and $v_l = x_n$)

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In Figure 1, the intuitionistic system that we call iIKN is an adaptation of the system described by Fitting in [3] to our notations and to the intuitionistic setting using the same method as for pure nested sequents, described in [11, 6]. In particular, we assign to every formula in the nested sequent a unique polarity: either *input* denoted by a $\bullet$-superscript or *output* denoted by a $\circ$-superscript.

A two-sided intuitionistic indexed nested sequent, denoted by Γ°, is a graph where each node contains a set of formulas, but with exactly one output formula in the whole graph. It can be written as a term:

$$\begin{array}{rrcccccc} & \Gamma^\circ & ::= & & & \Lambda^\bullet, A^\circ & \mid & \Lambda^\bullet, [^{v}\Gamma^\circ] \\ \text{where} & \Lambda^\bullet & ::= & \emptyset & \mid & \Lambda^\bullet, B^\bullet & \mid & \Lambda^\bullet, [^{u}\Lambda^\bullet] \end{array} \tag{4}$$

The bracketing represents the graph relation and the index on each bracket refers to a vertex.

Intuitively, once indexed, nested sequents are no longer trees, but any kind of *rooted* directed graphs[2] by identifying nodes carrying the same index. This is exemplified by the addition of the two structural rules tp and bc, called *teleportation* and *bracket-copy*, respectively, which are variants of the formula-contraction FC and the sequent-contraction SC of [3].

Finally, for a tuple (k, l, m, n) with $k, l, m, n \neq 0$, the rule $\mathsf{g_{klmn}}$ in Figure 2 is defined as in [3]. It must satisfy that $v_1 \ldots v_k$ and $x_1 \ldots x_n$ are fresh indexes which are pairwise distinct, except for the *confluence condition*: we always have $v_l = x_n$.

[1] This is the variant of IK first mentioned in [2] and [8] and studied in detail in [10].

[2] A *rooted* graph is a graph where one node is distinguished as the root and every node is reachable from it, i.e., the whole graph can be obtained as the minimal upward closure of this root for the edge relation.

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Let $\mathbb{G}$ be a set of 4-tuples, and G denote at the same time the correponding set of Scott-Lemmon axioms and of indexed nested sequent rules. Using the admissibility of the cut-rule in $\mathsf{iIKN}+\mathsf{G}$, we can prove the completeness of system $\mathsf{iKN}+\mathsf{G}$ with respect to system $\mathsf{IK}+\mathsf{G}$.

Theorem 1. *Let A be a modal formula. If A is provable in the Hilbert system $\mathsf{IK}+\mathsf{G}$, then the sequent A° is provable in the indexed nested sequent system $\mathsf{iIKN}+\mathsf{G}$.*

However, there are examples of theorems of $\mathsf{iIKN}+\mathsf{G}$ that are not theorems of $\mathsf{IK}+\mathsf{G}$, which means that the indexed nested sequent system is not sound with respect to the Hilbert axiomatisation using the axiom (3). There is already a simple counter-example (as mentioned in [10]) when one considers G containing only the axiom $\mathsf{g}_{1111}: \Diamond\Box A \supset \Box\Diamond A$.

Counter-example 1. *The formula:*

$$F = (\Diamond(\Box(a \vee b) \wedge \Diamond a) \wedge \Diamond(\Box(a \vee b) \wedge \Diamond b)) \supset \Diamond(\Diamond a \wedge \Diamond b) \tag{5}$$

is derivable in $\mathsf{iIKN}+\mathsf{g}_{1111}$, but is not a theorem of $\mathsf{IK}+\mathsf{g}_{1111}$.

Thus, the logic given by the Hilbert axiomatisation $\mathsf{IK}+\mathsf{G}$ and the one given by the indexed nested sequent system $\mathsf{iIKN}+\mathsf{G}$ actually differ in the intuitionistic case.

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In this section, we investigate the behaviour of system iIKN and its extension, regarding the Kripke semantics for intuitionistic modal logics. We can prove that any theorem of $\mathsf{iIKN}+\mathsf{G}$ is valid in every graph-consistent $\mathbb{G}$-model by showing that each rule of $\mathsf{iIKN}+\mathsf{G}$ is sound when interpreted in these models.

An *intuitionistic frame* $(W, R, \leq)$ is a non-empty set W of *worlds* and a binary relation $R \subseteq W \times W$, the *accessibility relation*, additionally equipped with a preorder $\leq$ on W s.t.:

(F1) For all $u, v, v' \in W$, if uRv and $v \leq v'$, there exists $u' \in W$ such that $u \leq u'$ and $u'Rv'$.

(F2) For all $u', u, v \in W$, if $u \leq u'$ and uRv, there exists $v' \in W$ such that $u'Rv'$ and $v \leq v'$.

An *intuitionistic model* $(W, R, \leq, V)$, is a intuitionistic frame together with a *valuation* function $V : W \to 2^{\mathcal{A}}$ mapping each world w to the set of propositional variables which are true in w, and that is monotone[3] with respect to $\leq$.

A $\mathbb{G}$-*model* is an intuitionistic model $\mathcal{M}$ where for each 4-tuple $(k, l, m, n) \in \mathbb{G}$, we have that: for all $w, u, v \in W$ with $wR^k u$ and $wR^m v$, there is a $z \in W$ such that $uR^l z$ and $vR^n z$[4] which is the "confluence" relation mentionned in the introduction.

Finally, we need to consider the notion of graph-consistency introduced by Simpson [10]. A intuitionistic model $\mathcal{M}$ is called *graph-consistent* if for any sequent Γ, given any homomorphism $h : \Gamma \mapsto \mathcal{M}$, any index w appearing in Γ, and any $w' \geq h(w)$, there exists another homomorphism $h' \geq h$ such that $h'(w) = w'$.

Theorem 2. *Let $\mathbb{G}$ be a set of 4-tuples and G be the corresponding set of indexed nested sequents rules. If a sequent Γ° is provable in $\mathsf{iIKN}+\mathsf{G}$ then it is valid in every graph-consistent intuitionistic $\mathbb{G}$-model.*

[3] $w \leq v$ implies $V(w) \subseteq V(v)$.

[4] We define the composition of two relations R, S on a set W as usual: $R \circ S = \{(w, v) \mid \exists u.\ (wRu \wedge uSv)\}$. R^n stands for R composed n times with itself.

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In the classical case, for a given set of 4-tuples $\mathbb{G}$, the logic given by Fitting's system [3] corresponds exactly to the logic axiomatised by the extension of the Hilbert system K with the Scott-Lemmon axioms G. As Fitting provides a semantical proof of soundness and completeness, one just needs to use the well-known correspondence between the logic $\mathsf{K}+\mathsf{G}$ and the semantics:

Theorem 3 (Lemmon and Scott [5])**.** *Let $\mathbb{G}$ be a set of 4-tuples. A formula is derivable in $\mathsf{K}+\mathsf{G}$ iff it is valid in all classical $\mathbb{G}$-models.*

In the intuitionistic case, the correspondence theory is much more tedious, and a lot of questions are still open. We do have Theorems 1 giving that every theorem of $\mathsf{IK}+\mathsf{G}$ is a theorem of $\mathsf{iIKN}+\mathsf{G}$, and Theorem 2 giving that every theorem of $\mathsf{iIKN}+\mathsf{G}$ is valid in graph-consistent $\mathbb{G}$-models, but there is no proper equivalent to Theorem 3 to "link" the two theorems into an actual soundness and completeness result for $\mathsf{iIKN}+\mathsf{G}$. As we have seen in Section 3, the first inclusion is strict, since the formula (5) is provable in $\mathsf{iIKN}+\mathsf{G}$, but not in $\mathsf{IK}+\mathsf{G}$. However, the strictness of the second inclusion is open.

Open question 1. *Is there a certain set of 4-tuples $\mathbb{G}$ such that there exists a formula that is valid in every intuitionistic graph-consistent $\mathbb{G}$-models, but that is not a theorem of $\mathsf{iIKN}+\mathsf{G}$, for G the set of rules corresponding to $\mathbb{G}$?*

To conclude, we can recall that for Simpson [10] there are two different (but equivalent) ways to define intuitionistic modal logics, either the natural deduction systems he proposes, or the extension of the standard translation for intuitionistic modal logics into first-order intuitionistic logic. Equivalence between the natural deduction systems and the Hilbert axiomatisations, or direct interpretation of the natural deduction systems in intuitionistic models are just side-results. He therefore sees their failure for the majority of logics not as a problem, but rather as another justification of the validity of the proof-theoretic approach. Thus, could the accurate definition for intuitionistic modal logics actually come from structural proof-theoretical studies rather than Hilbert axiomatisations or semantical considerations?

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