What are Combinatorial Proofs?

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In this talk I will push forward the claim that combinatorial proofs are canonical representations of proofs for logics that allow a clean separation between *additive* and *multiplicative* behaviour in a proof system. Usually, standard sequent calculi do not have this property. However, many deep inference proof systems allow such a separation via certain *strong decomposition theorems*. This suggests the following conjecture:

Conjecture 1. *A logic admits combinatorial proofs if and only if it has a deep inference proof system admitting a strong decomposition theorem.*

The table below gives an exhaustive list (as of time of this writing) of all logics for which combinatorial proofs exist together with the reference where combinatorial proofs have been described and reference where a proof of the corresponding decomposition theorem can be found.

logic	combinatorial proofs presented in	strong decomposition theorem proved in
MLL and MLL+mix	[Ret03]	trivial (there is no additive part)
MLL1	[Hug18]	trivial (there is no additive part)
ALL	[Hei11], [HH15]	trivial (there is no multiplicative part)
ALL1	[HHS18], [HHS19b]	trivial (there is no multiplicative part)
classical propositional logic	[Hug06a], [Hug06b]	[Str07]
classical first-order logic	[Hug19]	none, see Remark 2 below
intuitionistic propositional logic	[HHS19a]	none, see Remark 3 below
relevant logic	[AS19a]	[AS19a]
modal logics	[AS19b] (for the S4 tesseract)	[AS19b] (for the S4 tesseract)

Remark 2. For first-order classical logic many decompostion theorems exist, the simplest being Herbrand's theorem in the form of Gentzen's Midsequent theorem [Gen35]. More variants can be found in [Brü03] and [Ral19]. However, there is not yet a strong decompostion theorem that would be needed for establishing the relationship to combinatorial proofs.

Remark 3. For intuitionistic propositional logic no strong decomposition theorem has been published yet, but in can easily be proven using the combinatorial proofs of [HHS19a].

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