The Jacobson Radical of a Propositional Theory*†

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A Contribution to Ninety Years of Glivenko's Theorem

Abstract

Alongside the analogy between maximal ideals and complete theories, the Jacobson radical carries over from ideals of commutative rings to theories of propositional calculi. This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability, and the syntactical counterpart of which is Glivenko's Theorem. As a by-product we obtain a possible interpretation in logic of the axioms-as-rules conservation criterion for a multi-conclusion Scott-style entailment relation over a single-conclusion one.

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1 Introduction

Glivenko's theorem from 1929 says that if a propositional formula φ is provable in classical logic, then its double negation $\neg\neg\varphi$ is provable in intuitionistic logic. In 1933 Gödel extended this to predicate logic, which move required to admit on the intuitionistic side the scheme of double negation shift. With Gödel's and Gentzen's negative translation in place of double negation, both from 1933, one can even get by with minimal logic in place of intuitionistic logic. More than one related proof translation saw the light of the day, e.g. Kolmogorov's (1925) and Kuroda's (1951).

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Glivenko's theorem thus stood right at the beginning of a fundamental change of perspective: that classical logic can be embedded into intuitionistic or even minimal logic, rather than the latter being a limited version of the former. Together with the revision of Hilbert's Programme ascribed to Kreisel and Feferman, this has led to the much broader quest for the computational content of classical proofs, today culminating in agile areas such as dynamical algebra, formal topology, program extraction from proofs, proof analysis, proof mining and proof translations. The growing success of these approaches suggests that customary mathematics, with classical logic and set theory, might eventually prove to be much more constructive than widely thought.

In 1930 Tarski ascribed to Lindenbaum the theorem that in classical logic any given theory T equals the intersection of all the complete theories containing T. Its typical use for Gödel's Completeness Theorem aside, this Lindenbaum Lemma is one of several theorems from that period which describe the intersection of all the ideal objects extending a given concrete object. Those intersection theorems, in their full generality recognised as forms of the Axiom of Choice, are often put by contraposition as extension or separation theorems. Apart from Lindenbaum's, prominent cases are known by the names of Artin–Schreier, Hahn–Banach, Krull and Szpilrajn.

The case in algebra closest to Lindenbaum's Lemma, however, gained prominence only in 1945, when Jacobson pointed out the relevance of the intersection of all the maximal ideals of a given ring. In the present note we follow the analogy between maximal (proper) ideals and complete (consistent) theories to carry over the Jacobson radical from ideals of commutative rings to theories of propositional calculi (Section 5.2), where it turns out to coincide with the stable closure of a theory (Proposition 2, Corollary 2). This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability (Theorem 2), and the syntactical counterpart of which happens to be Glivenko's Theorem as recalled above (Theorem 3).

As a by-product we obtain a possible interpretation in logic (Theorem 4) of the axioms-as-rules conservation criterion (Theorem 1) for a multi-conclusion Scott-style entailment relation $\vdash$ over a single-conclusion one $\rhd$. This criterion has proved to be the common core of many a syntactical counterpart of a semantic conservation theorem corresponding to one of the aforementioned intersection theorems. Typically this kind of conservation means reduction to a special case characterised by additional axioms with (possibly empty) disjunctions in positive position. Applying the criterion means to eliminate the additional axioms for $\vdash$ by way of the corresponding disjunction elimination rules for $\rhd$. The latter equally suffice for proof practice, and have proved admissible in all mathematical instances yet considered. Our interpretation of the conservation criterion in propositional logic (Theorem 4) in fact is tantamount to Glivenko's Theorem (Theorem 3). As for the latter, disjunction elimination plays a central role in the proof of the former, together with some notorious features of (double) negation in intuitionistic logic (Lemma 1).

2 Preliminaries

Unless specified otherwise, we work in a suitable fragment of Aczel's *Constructive Zermelo–Fraenkel Set Theory* (**CZF**) [1–5] based on intuitionistic first-order predicate logic. While in general the concepts of this paper are elementary and the proofs are direct, we still pin down **CZF** as metatheory if only rather for convenience's sake; in fact much less might suffice. Likewise, when we occasionally need to invoke a fragment of the principle of Excluded Middle or even a form of the Axiom of Choice (AC), and thus go beyond **CZF**, we simply switch to **ZF** and **ZFC**, respectively, and indicate this accordingly.

For example, the *Restricted Law of Excluded Middle* (REM) is not a principle of **CZF**. This REM means $\varphi \vee \neg\varphi$ for every set-theoretic formula φ that is *bounded* in the sense that only set-bounded quantifiers of the types $\forall x \in y$ and $\exists x \in y$ occur in φ.

By a *finite* set we understand a set that can be written as $\{ a_1, \ldots, a_n \}$ for some $n \geq 0$. Given any set S, let $\mathrm{Pow}(S)$ (respectively, $\mathrm{Fin}(S)$) consist of the (finite) subsets of S. We refer to [81] for further provisos to carry over to the present note.[1]

We write $\rhd$ to denote (deducibility in) any intermediate logic in a propositional language S. The following intuitionistic properties of (double) negation are due to Brouwer [11, 12, 41, 96, 97]:

Lemma 1. *For any given intermediate logic* $\rhd$*,*

$$\frac{}{\neg\neg\neg\psi \rhd \neg\psi} \qquad \frac{\Gamma, \varphi \rhd \neg\psi}{\Gamma, \neg\neg\varphi \rhd \neg\psi} \qquad \frac{}{\rhd \neg\neg(\psi \vee \neg\psi)}$$

for every $\Gamma \in \mathrm{Fin}(S)$ *and all* $\varphi, \psi \in S$.

As is common in this context, negation is a defined connective: $\neg\varphi \equiv \varphi \to \bot$.

A *theory* of an intermediate logic $\rhd$ is a subset of the given propositional language S that is deductively closed with respect to $\rhd$. As usual, a theory T of $\rhd$ is *consistent* if $\bot \notin T$, which is to say that $T \neq S$; and a theory T of $\rhd$ is *complete* if

$$\forall \varphi \in S (T \ni \varphi \vee \neg T \ni \varphi) .$$

We say that a theory T of $\rhd$ is *stable* if

$$\forall \varphi \in S (T \ni \neg\neg\varphi \Rightarrow T \ni \varphi) .$$

Lemma 2. *The following are equivalent for every theory* T *of an intermediate logic* $\rhd$*:*

1. T *is deductively closed for classical logic.*
2. T *is stable.*
3. T *contains all instances of excluded middle* $\varphi \vee \neg\varphi$ *with* $\varphi \in S$.

Corollary 1. *If a theory of an intermediate logic* $\rhd$ *is complete, then it is stable.*

For a deeper discussion with earlier references we refer to [50] and [69, p. 27].

[1] For example, we deviate from the terminology prevalent in constructive mathematics and set theory [4, 5, 8, 9, 58, 63]: to reserve the term ‘finite’ to sets which are in *bijection* with $\{1, \ldots, n\}$ for a necessarily unique $n \geq 0$. Those exactly are the sets which are finite in our sense and are *discrete* too, i.e. have decidable equality [63].

3 Entailment Relations

Entailment relations, both in their single- and multi-conclusion variant, are at the heart of this note. We briefly recall the basic notions, to which end we closely follow [80, 81].

3.1 Consequence

Let S be a set and $\rhd \subseteq \mathrm{Pow}(S) \times S$. Once abstracted from the context of logical formulas, all but one of Tarski's axioms of *consequence* [95] can be put as

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{\forall b \in U (V \rhd b) \quad U \rhd a}{V \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{\exists U_0 \in \mathrm{Fin}(U)(U_0 \rhd a)}\ (\mathrm{A})$$

where $U, V \subseteq S$ and $a \in S$. These axioms also characterise a finitary covering or Stone covering in formal topology [84];[2] see further [17, 19, 66, 67, 86, 87]. The notion of consequence has allegedly been described first by Hertz [44–46]; see also [7, 54].

We do not employ the one of Tarski's axioms by which he required that S be countable. This aside, Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathrm{Pow}(S)$ of a relation $\rhd$ as above where

$$U^{\rhd} \equiv \{a \in S : U \rhd a\}.$$

3.2 Single-Conclusion Entailment

Rather than with Tarski's notion, we henceforth work with its restriction to finite subsets, that is, a *single-conclusion entailment relation*. This is a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ which satisfies

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{V \rhd b \quad V', b \rhd a}{V, V' \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\ (\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$. Our focus thus is on *finite* subsets of S, for which we henceforth reserve the letters $U, V, W, \ldots$; we also sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$ even if $n = 0$. Redefining

$$T^{\rhd} \equiv \{a \in S : \exists U \in \mathrm{Fin}(T)(U \rhd a)\} \tag{1}$$

for *arbitrary* subsets T of S gives back an algebraic closure operator on $\mathrm{Pow}(S)$; we occasionally write $T \rhd a$ to mean $a \in T^{\rhd}$. The single-conclusion entailment relations thus correspond exactly to the relations satisfying Tarski's axioms above.

[2] This is from where we have taken the symbol $\rhd$, used also [16, 98] to denote a 'consecution' [77].

3.3 Multi-Conclusion Entailment

Let S be a set and $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$. Scott's [91] axioms of entailment can be put as

$$\frac{U \between W}{U \vdash W}\ (\mathrm{R}) \qquad \frac{V \vdash W, b \quad V', b \vdash W'}{V, V' \vdash W, W'}\ (\mathrm{T}) \qquad \frac{U \vdash W}{U, U' \vdash W, W'}\ (\mathrm{M})$$

for finite $U, U', V, V', W, W' \subseteq S$ and $b \in S$, where $U \between W$ means that U and W have an element in common [86]. To be precise, any such $\vdash$ is a *multi-conclusion entailment relation*, where 'multi' includes 'empty'.

This fairly general notion of entailment has been introduced by Scott [90–92], building on Hertz's and Tarski's work (see above), and of course on Gentzen's sequent calculus [39, 40]. Shoesmith and Smiley [93] trace multi-conclusion entailment relations back to Carnap [13]. Before Scott, Lorenzen has developed analogous concepts formally [59–62]; he has even listed [60, pp. 84–5] counterparts of the axioms (R), (T) and (M) for single- and multi-conclusion entailment [27, 28].[3] As compared with Gentzen's and Lorenzen's approaches, Scott's entailment relation follow the contexts-as-sets paradigm, which has caused reservations [70, 71]. The relevance of the notion of entailment relation to point-free topology and constructive algebra has been pointed out in [14], and has been used very widely, e.g. in [20–22,24,25,30,33,72,78,83,89,99,102]. Consequence and entailment have further caught interest from various other angles [6, 35, 38, 47–49, 74, 88, 93, 103].

In practice, $\rhd$ and $\vdash$ are *inductively generated* by the axioms of the intended models, which procedure we here take for granted [14, 32]; see also [3, 76, 81, 82].

4 Conservation

Again following [80, 81], we sketch the concept of conservative extension of a multi-conclusion entailment relation $\vdash$ over a single-conclusion entailment relation $\rhd$ on the same set S. After that we extract from [79]—based on [18]—possible interpretations limited to classical logic.

4.1 Conservation in Syntax and Semantics

Let S be a set, and let $a, b, c, \ldots$ and $U, V, W, \ldots$ range over the elements of S and $\mathrm{Fin}(S)$, respectively. Given a multi-conclusion entailment relation $\vdash$ and a single-conclusion entailment relation $\rhd$ on the same set S, we throughout assume *Extension*:

$$\text{Ext} \qquad \frac{U \rhd a}{U \vdash a}$$

Of major interest to us is the converse, alias *Conservation*:

$$\text{Con} \qquad \frac{U \vdash a}{U \rhd a}$$

[3] Stefan Neuwirth has kindly pointed this out to us.

The *trace* of any given $\vdash$ is the single-conclusion entailment relation $\rhd_\vdash$ defined by

$$U \rhd_\vdash a \equiv U \vdash a\,,$$

for which Ext and Con are tantamount to $\rhd \subseteq \rhd_\vdash$ and $\rhd \supseteq \rhd_\vdash$, respectively.

An *arbitrary* subset P of S is a *model of* $\vdash$ if

$$P \supseteq U \Longrightarrow V \between P \quad \text{whenever } U \vdash V\,.$$

The notion of model carries over to single-conclusion $\rhd$ in the apparent manner, such that the *models of* $\rhd$ are exactly the $P \in \mathrm{Pow}(S)$ which are *closed* under $\rhd$, i.e. for which $P^{\rhd} = P$. Let $\mathrm{Mod}(\vdash)$ and $\mathrm{Mod}(\rhd)$ consist of the models of $\vdash$ and $\rhd$, respectively. By Extension, $\mathrm{Mod}(\vdash) \subseteq \mathrm{Mod}(\rhd)$, which in **ZFC** is equivalent to Extension [81, Lemma 9].

Now Con follows from the *Generalised Krull–Lindenbaum Lemma*, viz.

$$\text{GKL} \qquad \forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \Rightarrow a \in P) \implies U \rhd a\,,$$

the converse of which holds by Extension. Again by Extension, GKL implies the *Trace Completeness Theorem*, viz.

$$\text{TCT} \qquad \forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \Rightarrow a \in P) \implies U \vdash a\,,$$

the converse of which holds by the definition of a model of $\vdash$. This TCT is a fragment of AC that implies REM [81, Corollary 5].[4]

In **ZFC**, GKL and Con are equivalent [81, Theorem 6]. In **CZF** we can make this more precise:

Remark 1. *In the presence of* Ext*,* GKL *is equivalent to the conjunction of* Con *and* TCT.

In all, GKL is semantic conservation, and Con is its syntactical counterpart.

4.2 Conservation in Proof Practice

In proof practice, GKL is useful for reductions to special cases, by making possible to use $\vdash$ in proofs about $\rhd$, but GKL is of semantic nature, entails REM and requires some AC. In comparison, Con is equally sufficient for that kind of reduction, is syntactical and has elementary proofs. Many such cases are known in point-free topology such as locale theory and formal topology [14, 15, 20, 23, 64, 65]; in constructive algebra, especially with dynamical methods [26, 34, 55–58, 101, 104, 105]; and in the proof theory of order relations [70, 72]. Most of those cases concern algebra at large. But what about logic? One may think of Gentzen's classical multi-succedent sequent calculus as extending his intuitionistic single-succedent variant [39, 40, 69, 94]. As we will see, this thought goes in the right direction.

[4] The proof of [81, Proposition 4] goes equally through with TCT in place of full CT.

A typical situation is as follows: Let the single-conclusion entailment relation $\rhd$ on a set S be generated by axioms. Then the multi-conclusion entailment relation $\vdash$ on the same set S is generated by the axioms of $\rhd$, of course with $\vdash$ in place of $\rhd$, and by *additional axioms*

$$a_1, \ldots, a_k \vdash b_1, \ldots, b_\ell \tag{2}$$

where $k, \ell \geq 0$. In any such situation we say that $\vdash$ *extends* $\rhd$, and list the additional axioms if needed. This is legitimate inasmuch as if $\vdash$ extends $\rhd$, then Ext is satisfied. What about Con?

The following most versatile *conservation criterion* [80, 81], which in fact gathers together many of the cases of Con mentioned before, will also help to understand Con for logic:

Theorem 1. *Let* $\vdash$ *extend* $\rhd$ *with certain additional axioms of the form (2). Then* $\vdash$ *and* $\rhd$ *satisfy* Con *if and only if*

$$\frac{W, b_1 \rhd c \quad \ldots \quad W, b_\ell \rhd c}{W, a_1, \ldots, a_k \rhd c}$$

for any such additional axiom, all $c \in S$ *and every* $W \in \mathrm{Fin}(S)$.

This swiftly follows [81, Theorem 2] from a sandwich criterion for conservation given by Scott [91], and also is a corollary of cut elimination for entailment relations [82] as related to cut elimination in the presence of axioms [68].

Quite a few instances of GKL can be classified by the two cases named *Universal Krull* (UK) and *Universal Lindenbaum* (UL) in [81], for which S is a set with

UK : a distinguished element e of S and a binary operation $*$ on S

UL : a unary operation $\sim$ on S

The additional axioms for $\vdash$ extending $\rhd$ are

UK : $e \vdash \qquad a * b \vdash a, b$

UL : $a, {\sim}a \vdash \qquad \vdash a, {\sim}a$

where $a, b \in S$. The corresponding conservation criteria (Theorem 1) read

$$\text{UK}: \quad \frac{}{W, e \rhd c} \qquad \frac{W, a \rhd c \quad W, b \rhd c}{W, a * b \rhd c}$$

$$\text{UL}: \quad \frac{}{W, a, {\sim}a \rhd c} \qquad \frac{W, a \rhd c \quad W, {\sim}a \rhd c}{W \rhd c}$$

where $W \in \mathrm{Fin}(S)$ and $a, b, c \in S$.[5] We refer to [80, 81] for details and references.

[5]The criteria for UK have occurred [79] as 'e is *convincing* for $\rhd$' and '$\rhd$ satisfies *Encoding*'.

4.3 The Case of Classical Logic

Building upon [18], in [79] the instances of GKL in the cases UK and UL have been considered for the following data: S consists of the sentences of a logical language, $\rhd$ stands for deducibility with *classical* logic, e is absurdity $\bot$, the operator $*$ is disjunction $\vee$, and $\sim$ is negation $\neg$. While the models of $\rhd$ are the (classical) theories of $\rhd$, the models of $\vdash$ are the complete consistent theories of S. Hence GKL is Lindenbaum's Lemma [95], and Con is provable but little interesting, simply because $\rhd$ is classical logic already. Let's try to get more by relativising $\rhd$.

Now let $\rhd$ denote (deducibility in) an intermediate logic in a propositional language S; whence the models of $\rhd$ are the theories of $\rhd$.[6] Let $\vdash$ extend $\rhd$ with the following additional axioms:

$$\bot \vdash \qquad \vdash \varphi, \neg\varphi \quad (\varphi \in S)$$

The models of $\vdash$ are the complete consistent theories of $\rhd$, and the corresponding conservation criteria (Theorem 1) read

$$\frac{}{\Gamma, \bot \rhd \psi} \qquad \frac{\Gamma, \varphi \rhd \psi \quad \Gamma, \neg\varphi \rhd \psi}{\Gamma \rhd \psi}$$

with $\Gamma \in \mathrm{Fin}(S)$ and $\varphi, \psi \in S$. While the first criterion is provable, for any given intermediate logic $\rhd$, the second one amounts to $\rhd$ satisfying $\rhd \varphi \vee \neg\varphi$, which is to say that $\rhd$ be classical. Hence Con in this case simply means that adding $\vdash \varphi, \neg\varphi$ is equivalent to adding $\rhd \varphi \vee \neg\varphi$. This of course is well known and of little interest either. Can't we do better?

5 Jacobson Radicals

5.1 The Jacobson Radical in Algebra

Let $S = R$ be a commutative ring with 1, and let $\rhd$ stand for generation in R, i.e. $U \rhd a$ means that a is a linear combination with coefficients from R of the elements of U. A model of $\rhd$ is nothing but an *ideal* of R, i.e. a subset closed under linear combination. An ideal J of R is *proper* if $1 \notin J$, which is to say that $J \neq R$. A (proper) ideal J of R is a *maximal ideal* if

$$\forall r \in R (J \ni r \vee J, r \rhd 1) . \tag{3}$$

With this notation in place, the *Jacobson radical of an ideal J* can be defined as

$$\mathrm{Jac}(J) = \{a \in R : \forall b \in R (a, b \rhd 1 \Rightarrow J, b \rhd 1)\} . \tag{4}$$

Incidentally, we thus follow the first-order definition of the Jacobson radical for distributive lattices [10, 29, 53] rather than the more common one for commutative rings as given e.g. in [58],

$$\mathrm{Jac}(J) = \{a \in R : \forall b \in R \exists c \in R (1 - (1 - ab)c \in J)\} , \tag{5}$$

according to which any given $a \in R$ belongs to $\mathrm{Jac}(R)$ precisely when $1 - ab$ is invertible modulo J for every $b \in R$. In **ZFC**, the first-order definition we use (4) is anyway equivalent to the more customary second-order definition of the Jacobson radical [52]:

[6] For the related covering of formulas [31, 85], the saturated sets rather are the *complements* of the theories.

Proposition 1 (ZFC). *For every ideal J of R,*

$$\bigcap \mathrm{Max}_J(R) = \mathrm{Jac}(J)$$

where $\mathrm{Max}_J(R)$ *consists of the maximal (proper) ideals* $\mathfrak{m}$ *in R with* $J \subseteq \mathfrak{m}$.

Proof. Let $a \in \mathrm{Jac}(J)$, and let $\mathfrak{m}$ be a maximal ideal such that $\mathfrak{m} \supseteq J$. Either $\mathfrak{m} \ni a$ or $\mathfrak{m}, a \rhd 1$. In the former case we are done. In the latter case there is $b \in R$ such that $\mathfrak{m} \rhd b$ (in particular, $b \in \mathfrak{m}$) and $a, b \rhd 1$. Since $a \in \mathrm{Jac}(J)$, we get $J, b \rhd 1$. As $b \in \mathfrak{m}$, this implies $J, \mathfrak{m} \rhd 1$ and thus $\mathfrak{m} \rhd 1$. Hence $\mathfrak{m} = R$, by which $\mathfrak{m} \ni a$. Conversely, if $a \notin \mathrm{Jac}(J)$, then there exists $b \in R$ for which $a, b \rhd 1$ but $J, b \rhd 1$ fails, and thus $(J, b)^{\rhd}$ lacks a. Zorn's Lemma yields a maximal ideal $\mathfrak{m}$ that contains $(J, b)^{\rhd}$ yet misses a. □

It obviously is irrelevant whether the intersection ranges over the only improper ideal R as well. Although the proof above is necessarily similar to the one with (5) in place of (4), see e.g. [58], we have have given this one in detail because it will carry over to logic (Proposition 2).

5.2 The Jacobson Radical in Logic

Let again $\rhd$ be an intermediate logic in a propositional language S. That a (consistent) theory T of $\rhd$ be *complete* can equivalently be put as

$$\forall \varphi \in S (T \ni \varphi \vee T, \varphi \rhd \bot) . \tag{6}$$

Alongside the analogy between (3) and (6), we define the *Jacobson radical of a theory T*:

$$\begin{aligned} \mathrm{Jac}(T) &= \{\alpha \in S : \forall \beta \in S (\alpha, \beta \rhd \bot \Rightarrow T, \beta \rhd \bot)\} \\ &= \{\alpha \in S : \forall \beta \in S (\alpha \rhd \neg\beta \Rightarrow T \rhd \neg\beta)\}. \end{aligned}$$

Mutatis mutandis the proof of Proposition 1 proves the *Intermediate Lindenbaum Lemma*:

Theorem 2 (ZFC). *For every theory T of $\rhd$,*

$$\text{ILL} \qquad \bigcap \mathrm{Com}_T(S) = \mathrm{Jac}(T)$$

where $\mathrm{Com}_T(S)$ *consists of the complete (consistent) theories C in S with $T \subseteq C$.*

As for Proposition 1, it is irrelevant whether the intersection includes the only inconsistent theory S.

5.3 From Lindenbaum's Lemma to Glivenko's Theorem

We still consider theories T of an intermediate logic $\rhd$ in a propositional language S.

The left-hand side of ILL is as for the original Lindenbaum Lemma [95], and thus equals in **ZFC** the classical deductive closure of T. What about the right-hand side of ILL?

Proposition 2. *For every theory T of $\rhd$,*

$$\mathrm{Jac}(T) = \{\alpha \in S : T \ni \neg\neg\alpha\}.$$

Proof. Let $\alpha \in \mathrm{Jac}(T)$. Since $\alpha \rhd \neg\neg\alpha$, we have $T \rhd \neg\neg\alpha$. Conversely, let $\alpha \in S$ such that $T \ni \neg\neg\alpha$. If $\beta \in S$ is such that $\alpha \rhd \neg\beta$, then $\neg\neg\alpha \rhd \neg\beta$ and thus $T \rhd \neg\beta$. □

Corollary 2. $\mathrm{Jac}(T)$ *is the least* stable *theory in S which contains the given theory T of $\rhd$.*

In particular, if $\rhd$ is classical logic, then ILL is just the original Lindenbaum Lemma [95].

Now let $\rhd$ be intuitionistic logic $\rhd_i$, and write $\rhd_c$ for classical logic. In this case and by the above, ILL is the semantics of *Glivenko's Theorem* [42], well known as purely syntactical:

Theorem 3 (Glivenko 1929)**.** *For all $\Gamma \in \mathrm{Fin}(S)$ and $\varphi \in S$,*

$$\Gamma \rhd_c \varphi \Longrightarrow \Gamma \rhd_i \neg\neg\varphi.$$

Recent literature about Glivenko's Theorem proper includes [36, 43, 51, 73, 75].

5.4 Glivenko's Theorem as Syntactical Conservation

Let again $\rhd_i$ and $\rhd_c$ stand for intuitionistic and classical logic, respectively, in a propositional language S. For $\Gamma, \Delta \in \mathrm{Fin}(S)$ and $\varphi \in S$, define

$$\Gamma \rhd_g \varphi \equiv \Gamma \rhd_i \neg\neg\varphi \qquad \text{and} \qquad \Gamma \vdash_c \Delta \equiv \Gamma \rhd_c \bigvee \Delta,$$

which are a single- and a multi-conclusion entailment relation, respectively. By Glivenko's Theorem, $\rhd_g$ is nothing but $\rhd_c$, which equals the trace of $\vdash_c$. Hence $\vdash_c$ is a conservative extension of $\rhd_g$, but let's forget this for the sake of the subsequent argument.

Clearly, $\vdash_c$ extends $\rhd_g$ with the following additional axioms:

$$\bot \vdash_c \qquad \vdash_c \varphi, \neg\varphi \quad (\varphi \in S) \tag{7}$$

The corresponding conservation criteria (Theorem 1) read

$$\frac{}{\Gamma, \bot \rhd_g \psi} \qquad \frac{\Gamma, \varphi \rhd_g \psi \quad \Gamma, \neg\varphi \rhd_g \psi}{\Gamma \rhd_g \psi} \tag{8}$$

with $\Gamma \in \mathrm{Fin}(S)$ and $\varphi, \psi \in S$. The following assertion is actually equivalent to Glivenko's Theorem (Theorem 3); we still recall the proof if only to exhibit the role of disjunction elimination.

Theorem 4. *The extension $\vdash_c$ of $\rhd_g$ is conservative, that is,*

$$\text{Gli} \qquad \Gamma \vdash_c \varphi \Longrightarrow \Gamma \rhd_g \varphi$$

for all $\Gamma \in \mathrm{Fin}(S)$ and $\varphi \in S$.

Proof. Clearly, $\Gamma, \bot \rhd_i \neg\neg\psi$. If $\Gamma, \varphi \rhd_i \neg\neg\psi$ and $\Gamma, \neg\varphi \rhd_i \neg\neg\psi$, then $\Gamma, \varphi \vee \neg\varphi \rhd_i \neg\neg\psi$ by disjunction elimination. By Lemma 1 we get $\Gamma, \neg\neg(\varphi \vee \neg\varphi) \rhd_i \neg\neg\psi$ and thus $\Gamma \rhd_i \neg\neg\psi$. □

As the models of $\vdash_c$ are exactly the complete consistent theories, ILL for $T \equiv \Gamma^{\rhd_i}$ with $\Gamma \in \mathrm{Fin}(S)$ is to Gli just as GKL is to Con for $\vdash \equiv \vdash_c$ and $\rhd \equiv \rhd_g$. Although $\vdash_c$ equally extends $\rhd_i$ with the same additional axioms (7), and the first conservation criterion of (8) also holds for $\rhd_i$ in place of $\rhd_g$, this of course is not the case in general for the second one, e.g. if $\psi \equiv \varphi \vee \neg\varphi$.

References

[1] Peter Aczel. The type theoretic interpretation of constructive set theory. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 55–66. North-Holland, Amsterdam, 1978.

[2] Peter Aczel. The type theoretic interpretation of constructive set theory: choice principles. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 1–40. North-Holland, Amsterdam, 1982.

[3] Peter Aczel. The type theoretic interpretation of constructive set theory: inductive definitions. In *Logic, methodology and philosophy of science, VII (Salzburg, 1983)*, volume 114 of *Stud. Logic Found. Math.*, pages 17–49. North-Holland, Amsterdam, 1986.

[4] Peter Aczel and Michael Rathjen. Notes on constructive set theory. Technical report, Institut Mittag–Leffler, 2000. Report No. 40.

[5] Peter Aczel and Michael Rathjen. Constructive set theory. Book draft, 2010. URL: `https://www1.maths.leeds.ac.uk/~rathjen/book.pdf`.

[6] Arnon Avron. Simple consequence relations. *Inform. and Comput.*, 92:105–139, 1991.

[7] Jean-Yves Béziau. Les axiomes de Tarski. In Roger Pouivet and Manuel Resbuschi, editors, *La philosophie en Pologne 1919-1939*. Librairie Philosophique J. VRIN, Paris, 2006.

[8] Errett Bishop. *Foundations of Constructive Analysis*. McGraw-Hill, New York, 1967.

[9] Errett Bishop and Douglas Bridges. *Constructive Analysis*. Springer, 1985.

[10] Andreas Blass. Prime ideals yield almost maximal ideals. *Fund. Math.*, 127(1):57–66, 1987.

[11] L.E.J. Brouwer. De onbetrouwbaarheid der logische principes. *Tijdschrift voor Wijsbegeerte*, 2:152–158, 1908.

[12] L.E.J. Brouwer. Intuitionistische Zerlegung mathematischer Grundbegriffe. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 33:251–256, 1925.

[13] Rudolf Carnap. *Formalization of Logic*. Harvard University Press, Cambridge, MA, 1943.

[14] Jan Cederquist and Thierry Coquand. Entailment relations and distributive lattices. In Samuel R. Buss, Petr Hájek, and Pavel Pudlák, editors, *Logic Colloquium '98. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, Prague, Czech Republic, August 9–15, 1998*, volume 13 of *Lect. Notes Logic*, pages 127–139. A. K. Peters, Natick, MA, 2000.

[15] Jan Cederquist, Thierry Coquand, and Sara Negri. The Hahn–Banach theorem in type theory. In G. Sambin and J.M. Smith, editors, *Twenty-Five Years of Constructive Type Theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 57–72. Oxford University Press, New York, 1998.

[16] Petr Cintula and Noguera Carles. The proof by cases property and its variants in structural consequence relations. *Studia Logica*, 101(4):713–747, 2013.

[17] Francesco Ciraulo, Maria Emilia Maietti, and Giovanni Sambin. Convergence in formal topology: a unifying notion. *J. Log. Anal.*, 5, 2013.

[18] Francesco Ciraulo, Davide Rinaldi, and Peter Schuster. Lindenbaum's lemma via open induction. In R. Kahle, T. Strahm, and T. Studer, editors, *Advances in Proof Theory*,

volume 28 of *Progress in Computer Science and Applied Logic*, pages 65–77. Springer International Publishing Switzerland, Cham, 2016.

[19] Francesco Ciraulo and Giovanni Sambin. Finitary formal topologies and Stone's representation theorem. *Theoret. Comput. Sci.*, 405(1–2):11–23, 2008.

[20] Thierry Coquand. A direct proof of the localic Hahn-Banach theorem, 2000. URL: `http://www.cse.chalmers.se/~coquand/formal.html`.

[21] Thierry Coquand. Lewis Carroll, Gentzen and entailment relations, 2000. URL: `http://www.cse.chalmers.se/~coquand/formal.html`.

[22] Thierry Coquand. About Stone's notion of spectrum. *J. Pure Appl. Algebra*, 197(1–3):141–158, 2005.

[23] Thierry Coquand. Geometric Hahn–Banach theorem. *Math. Proc. Cambridge Philos. Soc.*, 140:313–315, 2006.

[24] Thierry Coquand. Space of valuations. *Ann. Pure Appl. Logic*, 157:97–109, 2009.

[25] Thierry Coquand and Henri Lombardi. Hidden constructions in abstract algebra: Krull dimension of distributive lattices and commutative rings. In M. Fontana, S.-E. Kabbaj, and S. Wiegand, editors, *Commutative Ring Theory and Applications*, volume 231 of *Lect. Notes Pure Appl. Mathematics*, pages 477–499, Reading, MA, 2002. Addison-Wesley.

[26] Thierry Coquand and Henri Lombardi. A logical approach to abstract algebra. *Math. Structures Comput. Sci.*, 16:885–900, 2006.

[27] Thierry Coquand, Henri Lombardi, and Stefan Neuwirth. Lorenzen's theory of divisibility. Preprint, 2016.

[28] Thierry Coquand, Henri Lombardi, and Stefan Neuwirth. Lattice-ordered groups generated by an ordered group and regular systems of ideals. *Rocky Mountain J. Math.*, 2019. Preprint.

[29] Thierry Coquand, Henri Lombardi, and Claude Quitté. Dimension de Heitmann des treillis distributifs et des anneaux commutatifs. *Publications Mathématiques de Besançon. Algèbre et Théorie des Nombres*, pages 57–100, 2006.

[30] Thierry Coquand and Henrik Persson. Valuations and Dedekind's Prague theorem. *J. Pure Appl. Algebra*, 155(2–3):121–129, 2001.

[31] Thierry Coquand, Sara Sadocco, Giovanni Sambin, and Jan M. Smith. Formal topologies on the set of first-order formulae. *J. Symb. Log.*, 65(3):1183–1192, 2000.

[32] Thierry Coquand, Giovanni Sambin, Jan Smith, and Silvio Valentini. Inductively generated formal topologies. *Ann. Pure Appl. Logic*, 124:71–106, 2003.

[33] Thierry Coquand and Guo-Qiang Zhang. Sequents, frames, and completeness. In Peter G. Clote and Helmut Schwichtenberg, editors, *Computer Science Logic (Fischbachau, 2000)*, volume 1862 of *Lecture Notes in Comput. Sci.*, pages 277–291. Springer, Berlin, 2000.

[34] Michel Coste, Henri Lombardi, and Marie-Françoise Roy. Dynamical method in algebra: Effective Nullstellensätze. *Ann. Pure Appl. Logic*, 111(3):203–256, 2001.

[35] Kosta Došen. On passing from singular to plural consequences. In Ewa Orlowska, editor, *Logic at Work: Essays Dedicated to the Memory of Helena Rasiowa*, volume 24 of *Stud. Fuzziness Soft Comput.*, pages 533–547. Physica, Heidelberg, 1999.

[36] Christian Espíndola. A short proof of Glivenko theorems for intermediate predicate logics. *Arch. Math. Logic*, 52(7-8):823–826, 2013.

[37] Giulio Fellin. The Jacobson Radical: from Algebra to Logic. Master's thesis. Università di Verona, Dipartimento di Informatica, 2018.

[38] Dov M. Gabbay. *Semantical investigations in Heyting's intuitionistic logic*, volume 148 of *Synthese Library*. D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981.

[39] Gerhard Gentzen. Untersuchungen über das logische Schließen I. *Math. Z.*, 39:176–210, 1934.

[40] Gerhard Gentzen. Untersuchungen über das logische Schließen II. *Math. Z.*, 39:405–431, 1934.

[41] Valery Glivenko. Sur la logique de M. Brouwer. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, 14:225–228, 1928.

[42] Valery Glivenko. Sur quelques points de la Logique de M. Brouwer. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, 15:183–188, 1929.

[43] Giulio Guerrieri and Alberto Naibo. Postponement of raa and Glivenko's theorem, revisited. *Studia Logica*, 107(1):109–144, 2019.

[44] Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. I. Teil. Sätze ersten Grades. *Math. Ann.*, 87(3):246–269, 1922.

[45] Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. II. Teil. Sätze höheren Grades. *Math. Ann.*, 89(1):76–102, 1923.

[46] Paul Hertz. Über Axiomensysteme für beliebige Satzsysteme. *Math. Ann.*, 101(1):457–514, 1929.

[47] Lloyd Humberstone. On a conservative extension argument of Dana Scott. *Log. J. IGPL*, 19:241–288, 2011.

[48] Lloyd Humberstone. Dana Scott's work with generalized consequence relations. In Jean-Yves Béziau, editor, *Universal Logic: An Anthology. From Paul Hertz to Dov Gabbay*, Stud. Univers. Log., pages 263–279. Birkhäuser, Basel, 2012.

[49] Rosalie Iemhoff. Consequence relations and admissible rules. *J. Philosophical Logic*, 45(3):327–348, 2016.

[50] Hajime Ishihara. Classical propositional logic and decidability of variables in intuitionistic propositional logic. *Log. Methods Comput. Sci.*, 10(3):3:1, 7, 2014. URL: `https://doi.org/10.2168/LMCS-10(3:1)2014`, `doi:10.2168/LMCS-10(3:1)2014`.

[51] Hajime Ishihara and Helmut Schwichtenberg. Embedding classical in minimal implicational logic. *MLQ Math. Log. Q.*, 62(1-2):94–101, 2016.

[52] Nathan Jacobson. The radical and semi-simplicity for arbitrary rings. *Amer. J. Math.*, 67(2):300–320, 1945. `doi:10.2307/2371731`.

[53] Peter Johnstone. Almost maximal ideals. *Fund. Math.*, 123(3):197–209, 1984.

[54] Javier Legris. Paul Hertz and the origins of structural reasoning. In Jean-Yves Béziau, editor, *Universal Logic: An Anthology. From Paul Hertz to Dov Gabbay*, Studies in Universal Logic, pages 3–10. Birkhäuser, Basel.

[55] Henri Lombardi. Le contenu constructif d'un principe local-global avec une application à la structure d'un module projectif de type fini. *Publications Mathématiques de Besançon. Algèbre et Théorie des Nombres*, 1997. Fascicule 94–95 & 95–96.

[56] Henri Lombardi. Relecture constructive de la théorie d'Artin–Schreier. *Ann. Pure Appl. Logic*, 91:59–92, 1998.

[57] Henri Lombardi. Algèbre dynamique, espaces topologiques sans points et programme de Hilbert. *Ann. Pure Appl. Logic*, 137:256–290, 2006.

[58] Henri Lombardi and Claude Quitté. *Commutative Algebra: Constructive Methods: Finite Projective Modules*. Springer Netherlands, Dordrecht, 2015.

[59] Paul Lorenzen. Über halbgeordnete Gruppen. *Math. Z.*, 52(1):483–526, 1950.

[60] Paul Lorenzen. Algebraische und logistische Untersuchungen über freie Verbände. *J. Symb. Logic*, 16(2):81–106, 1951.

[61] Paul Lorenzen. Teilbarkeitstheorie in Bereichen. *Math. Z.*, 55(3):269–275, 1952.

[62] Paul Lorenzen. Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen. *Math. Z.*, 58(1):15–24, 1953.

[63] Ray Mines, Fred Richman, and Wim Ruitenburg. *A Course in Constructive Algebra*. Springer, New York, 1988. Universitext.

[64] Christopher J. Mulvey and Joan Wick-Pelletier. The dual locale of a seminormed space. *Cah. Topol. Géom. Différ. Catég.*, 23(1):73–92, 1982.

[65] Christopher J. Mulvey and Joan Wick-Pelletier. A globalization of the Hahn–Banach theorem. *Adv. Math.*, 89:1–59, 1991.

[66] Sara Negri. Stone bases alias the constructive content of Stone representation. In Aldo Ursini and Paolo Aglianò, editors, *Logic and algebra. Proceedings of the international conference dedicated to the memory of Roberto Magari, April 26–30, 1994, Pontignano, Italy*, volume 180 of *Lecture Notes in Pure and Applied Mathematics*, pages 617–636. Marcel Dekker, New York, 1996.

[67] Sara Negri. Continuous domains as formal spaces. *Math. Structures Comput. Sci.*, 12(1):19–52, 2002.

[68] Sara Negri and Jan von Plato. Cut elimination in the presence of axioms. *Bull. Symb. Log.*, 4(4):418–435, 1998.

[69] Sara Negri and Jan von Plato. *Structural Proof Theory*. Cambridge University Press, Cambridge, 2001.

[70] Sara Negri and Jan von Plato. *Proof Analysis. A Contribution to Hilbert's Last Problem*. Cambridge University Press, Cambridge, 2011.

[71] Sara Negri and Jan von Plato. Cut elimination in sequent calculi with implicit contraction, with a conjecture on the origin of Gentzen's altitude line construction. In D. Probst and P. Schuster, editors, *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, volume 6 of *Ontos Mathematical Logic*, pages 269–290. Walter de Gruyter, Berlin, 2016.

[72] Sara Negri, Jan von Plato, and Thierry Coquand. Proof-theoretical analysis of order relations. *Arch. Math. Logic*, 43:297–309, 2004.

[73] Hiroakira Ono. Glivenko theorems revisited. *Ann. Pure Appl. Logic*, 161(2):246–250, 2009.

[74] Gillman Payette and Peter K. Schotch. Remarks on the Scott–Lindenbaum Theorem. *Studia Logica*, 102(5):1003–1020, 2014.

[75] Luiz Carlos Pereira and Edward Hermann Haeusler. On constructive fragments of classical logic. In *Dag Prawitz on proofs and meaning*, volume 7 of *Outst. Contrib. Log.*, pages 281–292. Springer, Cham, 2015.

[76] Michael Rathjen. Generalized inductive definitions in constructive set theory. In Laura Crosilla and Peter Schuster, editors, *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics*, volume 48 of *Oxford Logic Guides*, chapter 16. Clarendon Press, Oxford, 2005.

[77] Greg Restall. *An Introduction to Substructural Logics*. Routledge, London, 2000.

[78] Davide Rinaldi. *Formal Methods in the Theories of Rings and Domains*. Doctoral dissertation, Universität München, 2014.

[79] Davide Rinaldi and Peter Schuster. A universal Krull–Lindenbaum theorem. *J. Pure Appl. Algebra*, 220:3207–3232, 2016.

[80] Davide Rinaldi, Peter Schuster, and Daniel Wessel. Eliminating disjunctions by disjunction elimination. *Bull. Symb. Logic*, 23(2):181–200, 2017.

[81] Davide Rinaldi, Peter Schuster, and Daniel Wessel. Eliminating disjunctions by disjunction elimination. *Indag. Math. (N.S.)*, 29(1):226–259, 2018.

[82] Davide Rinaldi and Daniel Wessel. Cut elimination for entailment relations. *Arch. Math. Log.*, 2018. URL: `https://doi.org/10.1007/s00153-018-0653-0`.

[83] Davide Rinaldi and Daniel Wessel. Extension by conservation. Sikorski's theorem. *Log. Methods Comput. Sci.*, 14(4:8):1–17, 2018.

[84] Giovanni Sambin. Intuitionistic formal spaces—a first communication. In D. Skordev, editor, *Mathematical Logic and its Applications, Proc. Adv. Internat. Summer School Conf., Druzhba, Bulgaria, 1986*, pages 187–204. Plenum, New York, 1987.

[85] Giovanni Sambin. Pretopologies and completeness proofs. *J. Symb. Log.*, 60(3):861–878, 1995.

[86] Giovanni Sambin. Some points in formal topology. *Theoret. Comput. Sci.*, 305(1–3):347–408, 2003.

[87] Giovanni Sambin. *The Basic Picture. Structures for Constructive Topology*. Oxford Logic Guides. Clarendon Press, Oxford, forthcoming.

[88] Tor Sandqvist. Preservation of structural properties in intuitionistic extensions of an inference relation. *Bull. Symb. Log.*, 24(3):291–305, 2018.

[89] Konstantin Schlagbauer, Peter Schuster, and Daniel Wessel. Der Satz von Hahn–Banach per Disjunktionselimination. *Confluentes Math.*, 201? Forthcoming.

[90] Dana Scott. On engendering an illusion of understanding. *J. Philos.*, 68:787–807, 1971.

[91] Dana Scott. Completeness and axiomatizability in many-valued logic. In Leon Henkin, John Addison, C.C. Chang, William Craig, Dana Scott, and Robert Vaught, editors, *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 411–435. Amer. Math. Soc., Providence, RI, 1974.

[92] Dana S. Scott. Background to formalization. In Hugues Leblanc, editor, *Truth, syntax and modality (Proc. Conf. Alternative Semantics, Temple Univ., Philadelphia, Pa., 1970)*, pages 244–273. Studies in Logic and the Foundations of Math., Vol. 68. North-Holland, Amsterdam, 1973.

[93] D. J. Shoesmith and T. J. Smiley. *Multiple-Conclusion Logic*. Cambridge University Press, Cambridge, 1978.

[94] Gaisi Takeuti. *Proof Theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, second edition, 1987. With an appendix containing contributions by Georg Kreisel, Wolfram Pohlers, Stephen G. Simpson and Solomon Feferman.

[95] Alfred Tarski. Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. I. *Monatsh. Math. Phys.*, 37:361–404, 1930.

[96] Anne Sjerp Troelstra. On the early history of intuitionistic logic. In Petio Petrov Petkov, editor, *Mathematical Logic*, pages 3–17. Plenum Press, New York, 1990.

[97] Mark van Atten and Göran Sundholm. L.E.J. Brouwer's 'Unreliability of the Logical Principles': A new translation, with an introduction. *History and Philosophy of Logic*, 38(1):24–47, 2017.

[98] San-min Wang and Petr Cintula. Logics with disjunction and proof by cases. *Arch. Math. Logic*, 47(5):435–446, 2008.

[99] Daniel Wessel. Ordering groups constructively. *Comm. Algebra*, 201? Forthcoming.

[100] Daniel Wessel. *Choice, extension, conservation. From transfinite to finite proof methods in abstract algebra*. Phd thesis, Università degli Studi di Trento, 2018.

[101] Daniel Wessel. A note on connected reduced rings. *J. Comm. Algebra*, 2019. To appear.

[102] Daniel Wessel. Point-free spectra of linear spreads. In S. Centrone, S. Negri, D. Sarikaya, and P. Schuster, editors, *Mathesis Universalis, Computability and Proof*, Synthese Library. Springer, forthcoming.

[103] Ryszard Wójcicki. *Theory of logical calculi. Basic theory of consequence operations*, volume 199 of *Synthese Library*. Kluwer Academic Publishers Group, Dordrecht, 1988.

[104] Ihsen Yengui. Making the use of maximal ideals constructive. *Theoret. Comput. Sci.*, 392:174–178, 2008.

[105] Ihsen Yengui. *Constructive Commutative Algebra. Projective Modules over Polynomial Rings and Dynamical Gröbner Bases*, volume 2138 of *Lecture Notes in Mathematics*. Springer, Cham, 2015.